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THE LIFE AND WORKS OF A. K. ERLANG

BY

E. BROCKMEYER, H. L. HALSTRØM
AND ARNE JENSEN

COPENHAGEN
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PREFACE

The Directors of the Copenhagen Telephone Company wish to publish the present work in memory of the late scientific collaborator of the Company, Mr. *A. K. Erlang*, M. A., as "Le comité consultatif international des communications téléphoniques à grande distance" (C. C. I. F.) at its plenary meeting at Montreux in October, 1946, has decided that the name of Erlang shall be connected with the International Unit of Telephone Traffic. The publishing is further occasioned by the fact that Mr. Erlang would have attained the age of 70 on the 1st January, 1948.

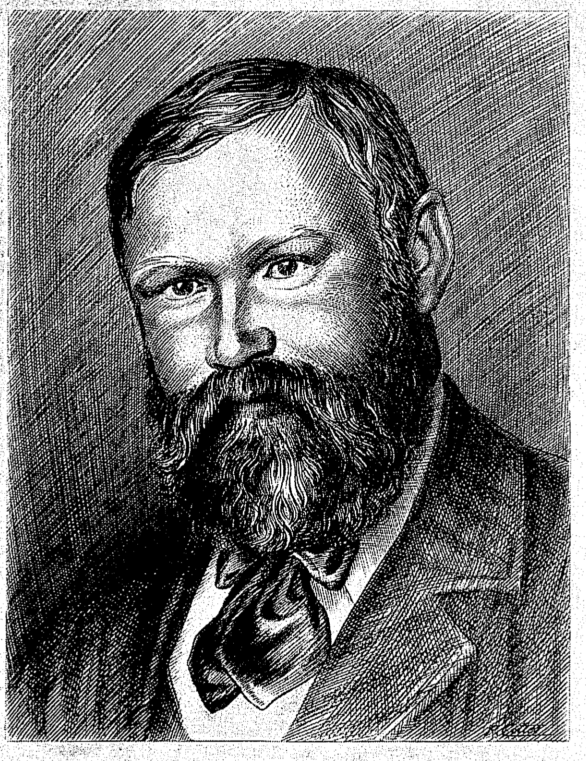
The publication of this work is intended to meet a desire, expressed from various quarters, of a complete edition of A. K. Erlang's principal works which have been available hitherto in the form of articles in Danish and foreign journals only. The present collection of Erlang's works has been edited by 3 members of the staff of the Telephone Company, *viz.* Mr. *E. Brockmeyer* and Mr. *H. L. Halstrøm*, telephone engineers, MM. Sc., and Mr. *Arne Jensen*, actuary. Furthermore, Mr. Brockmeyer and Mr. Halstrøm have written a biography of Erlang and a commented survey of his works, and Mr. Arne Jensen has written a paper, in which Erlang's works on the theory of probabilities are recapitulated and elucidated on the basis of the theory of stochastic processes. Mr. *Poul Reppien*, sworn translator, has performed the translation into English.

The Directors of the Company wish to extend their best thanks to the "*Academy of Technical Sciences*" for its kindness in including this work among its "Transactions".

It is their hope that this book may serve to make A. K. Erlang's important scientific achievements — especially those pertaining to the theory of telephone traffic — known and appreciated to an even greater extent than before.

The Copenhagen Telephone Company, Ltd.

G. Irming.



AGNER KRARUP ERLANG
1878 - 1929

This portrait of *A. K. Erlang*, which originally appeared in the *Journal des Télécommunications*, December 1947, is here reproduced by kind permission of *Le Bureau de l'Union Internationale des Télécommunications*.

THE LIFE OF A. K. ERLANG

By E. BROCKMEYER and H. L. HALSTRØM[1]).

"Bene qui latuit, bene vixit".
Ovid: Tristia, 3, 4, 25.

Agner Krarup Erlang was born on Tuesday, the 1st January, 1878, at Lønborg, a neighbouring village to the small town of Tarm situated south of the Skern rivulet in Jutland.

His father, *Hans Nielsen Erlang*, was the worthy parish clerk of the village, his official title being that of "schoolmaster and precentor". H. N. Erlang was born in South Jutland; nothing much is known about his family, but the name of Erlang is believed to be a corruption — for which German clergymen probably are responsible — of the name of Erlandsen. He had received a good education at a teachers' training college at Jelling, under the influence of *H. J. M. Svendsen*, a prominent member of the Danish folk high school, who was head of the college at the time. This college was preferred by a great number of Danish-minded students from Slesvig who did not want to receive their training at the strongly German-influenced college at Tønder.

A. K. Erlang's mother's family, on the other hand, is easier to trace. His mother was *Magdalene Krarup*, of the well-known ecclesiastical family. One of his maternal ancestors was the prominent mathematician *Thomas Fincke*, a contemporary of the great astronomer, *Tycho Brahe*; Fincke's descendants through several hundred years were holders of chairs in the University of Copenhagen. The Krarup family was, furthermore, related to the famous poet and religious leader, *N. F. S. Grundtvig*, the latter being a first cousin to Magdalene Krarup's grandmother. It was a tradition in the Krarup family that all sons should become clergymen and all daughters, clergymen's wives; and so it was a little unusual when young Magdalene chose to marry a plain village schoolmaster. But as the family got to know H. N. Erlang, they soon learnt to like and respect him very much.

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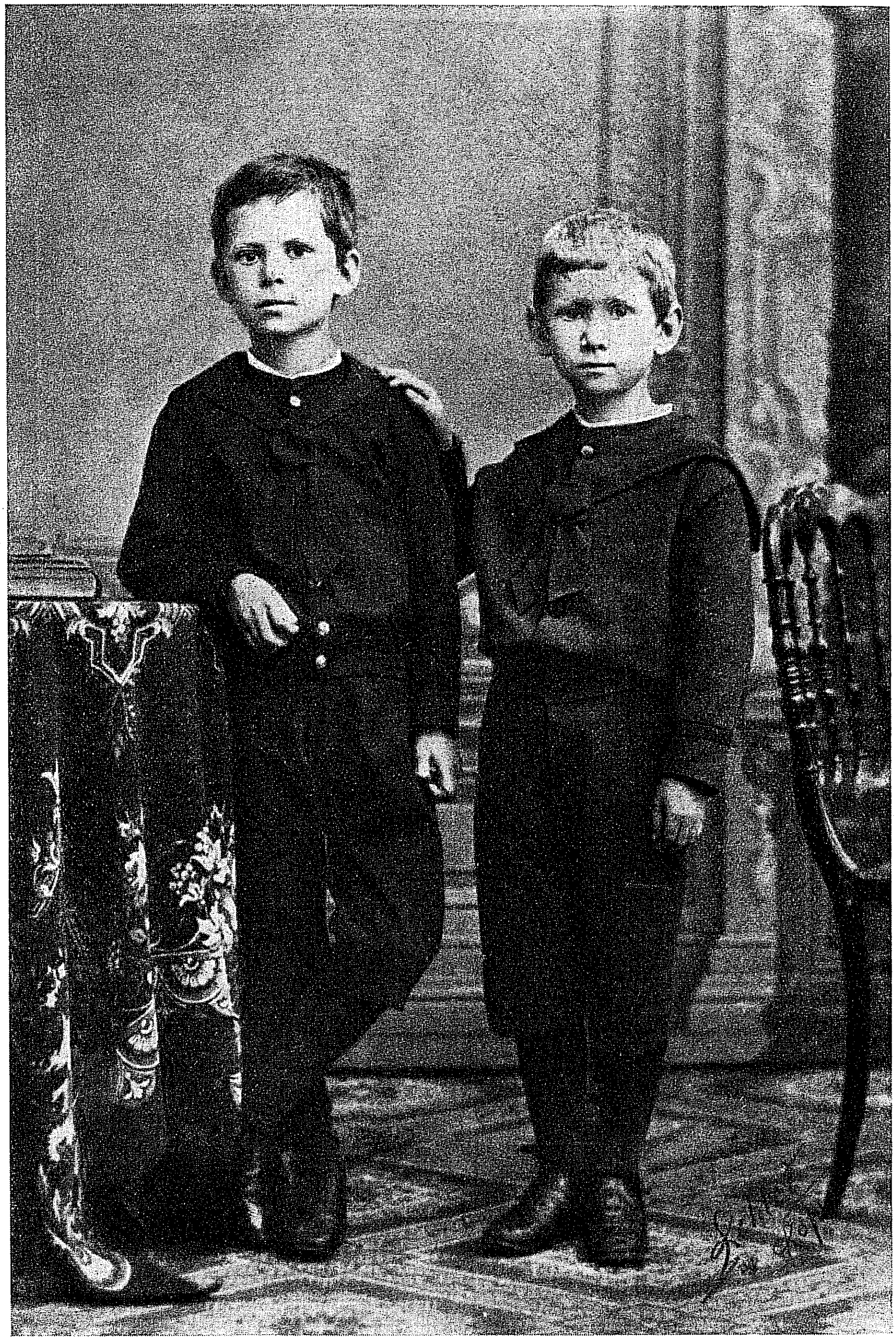
[1]) The authors are greatly indebted to the late Mr. *F. K. Erlang* and to Miss *I. Erlang* for their kindness in supplying valuable biographical information.

Agner grew up at home together with his brother, *Frederik*, by two years his senior, and his two younger sisters, *Marie* and *Ingeborg*, in not too easy circumstances. The schoolmaster's salary was very small, but he was a clever and intelligent man with an unusual talent for economizing, and his wife did not mind working hard. Therefore the family lived happily together in spite of their small income, and the children had a happy childhood and a proper upbringing. One of the father's principles was that the children should have all the food they needed, but the food must be plain, and preferably milk food; he had also decided that all milk should be carefully and thoroughly boiled so as to kill any disease-carrying germs. Although inconsistent with modern methods of preserving vitamins, this procedure was unquestionably a very sensible precaution at the time, considering the then wide-spread cattle tuberculosis.

Agner soon proved to be quick of apprehension, and he had a good memory, too; he was a quiet and peaceable boy who preferred reading to playing with the other boys. In the evenings, he and his elder brother would often share the reading of a book between them, the usual procedure being that brother Frederik would read it in the approved manner, while Agner, sitting opposite to him at the table, would read the book upside down. They got along well, as he soon learnt to read just as well in this new manner as Frederik did in his. Agner's favourite subject at that particular time was astronomy, and it was a great help that grandfather Krarup, the old vicar, had had the same hobby and had left several volumes on astronomy when he died. But Agner was not only interested in scientifical studies; the art of verse-making had also aroused his interest, and he wrote a good many poems on astronomical subjects. His poems, however, did not distinguish themselves so much by poetic flight as by strict logic. He wrote, for instance, a poem about the discovery of the planet Neptune, the first few lines of which, reproduced in English, read as follows:

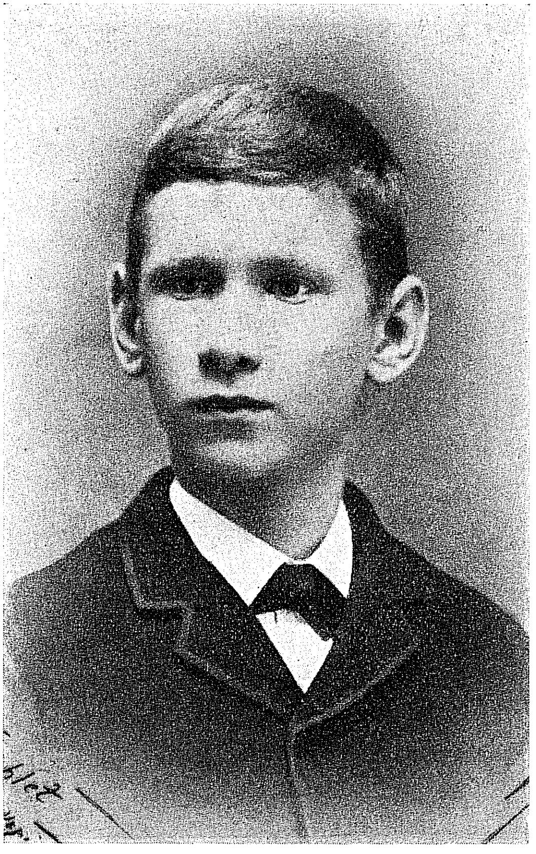
"Leverrier wrote to Galle a letter,
Saying, 'Gaze at the sky you had better,
For soon up there, may be,
A planet you will see!'"

Agner spent his early school-days at his father's schoolhouse together with his brother and sisters, and when he had finished his elementary education at the village school, he stayed at home to be coached for the "Præliminæreksamen" (a sort of lower school certificate examination), partly by his father and partly, or rather chiefly, by *P. J. Pedersen*, an assistant teacher who had just been appointed to help the schoolmaster. P. J. Pedersen had



Agner Erlang at the age of 8 (right) and his brother Frederik.

graduated from a teachers' training college and was an excellent teacher; his teaching came to be of consequence to Agner Erlang's development and career. Several years later he became Mayor of Copenhagen. — In the summer of 1892, when it was fairly safe to presume that a sufficient amount of knowledge had been imparted to them, Agner and his brother



A. K. Erlang at the age of 15.

Frederik were sent to Copenhagen to enter as privately coached examinees for the "Præliminæreksamen" which was held at the University. Agner was then only 14 years old, which was below the prescribed age limit; it was therefore necessary to apply for a special entrance permission. The permission was granted, however, and Agner passed his examination with distinction.

After that, Agner returned to Lønborg and became assistant teacher at his father's school for a period of two years; at the same time he was taught

French by a teacher at the "Tarm Realskole" and Latin by the vicar of a neighbouring parish. When Agner reached the age of 16 his father thought that he ought to continue his studies and, if possible, pass the "Studentereksamen" (the University entrance examination). But money was scarce, and so Agner's father had to find out how much this might be expected to cost him. He wrote, therefore, to *M. Funch*, the county clerk of Hillerød[1]), whose wife was born Krarup, for information and advice, as their son had recently passed the "Studentereksamen". The county clerk wrote back that he had always felt indebted to the Krarup family, and would Erlang please permit him to take young Agner into his home — by way of repaying a small part of this debt — while he prepared for his examination at the Frederiksborg Grammar-School? This generous offer helped to overcome the financial difficulties in connexion with Agner Erlang's education, and he spent a happy time with the county clerk at Hillerød until he, in 1896, passed his "Studentereksamen" with distinction.

The Frederiksborg Grammar-School had the privilege of two scholarships tenable at "Regensen"[2]), and Erlang attained one of these. He now lived at "Regensen" while he studied mathematics and exact natural sciences at the University of Copenhagen. His mathematical education was greatly influenced by Prof. *H. G. Zeuthen*'s and Prof. *C. Juel*'s lectures, and all his life he maintained and cultivated his interest in geometrical problems. He finished his university studies in January, 1901, by taking the degree of candidatus magisterii (M.A.) with mathematics as principal subject and astronomy, physics, and chemistry as secondary subjects.

For some years Erlang worked as a teacher at various schools, such as: "Gammelholms Latin- & Realskole", "Femmers Kvindeseminarium" and "Lang & Hjorts Kursus" in Copenhagen, and "Vamdrup Realskole" in South Jutland. He proved to be in possession of excellent pedagogical qualities, even though his natural predilection was for scientific research rather than for teaching.

During this period, Erlang used to spend his leisure hours with a circle of young university people at "Studenterhjemmet", a Christian students' association in Copenhagen. Sympathizing with the Christian students' movement, he was for a time a member of the management of "Studenterhjemmet". He was a peaceable, not particularly sociable man who preferred to walk about as an interested spectator; this had inspired his friends to nickname him "The Private Person". Among the acquaintances he made at "Studenterhjemmet" was *H. C. Nybølle* who later was appointed professor of statistics in the University of Copenhagen. Erlang and Ny-

[1]) Market town, North Seeland, with ancient royal castle of Frederiksborg.

[2]) "Regensen" is an ancient college in Copenhagen providing free lodgings for deserving university students; it was founded in 1623 by King Christian IV.

bølle became friends for life, and they helped each other with their scientific work. Later, Nybølle's sister married A. K. Erlang's brother, *F. K. Erlang*.

In conjunction with his work as a teacher, Erlang diligently continued his studies of mathematics and natural sciences; thus, he managed to find time for attempting the mathematical prize essay of the University for 1902-1903, on Huygens' solutions of infinitesimal problems, for which attempt he was rewarded with an "accessit" in 1904. He had then already taken up the study of the theory of probabilities which later came to be the subject of his principal works.

In those years — and later — Erlang spent several of his summer vacations abroad, frequently accompanied by his brother, and a few times accompanied by *L. Christensen*, his fellow collegian who had become headmaster of the Cathedral School of Aarhus. Thus, he went to Sweden, England, Germany, and France several times. On these journeys he cultivated his many-sided interests, visiting art galleries and libraries, and he is said to have been an uncommonly interesting and pleasant travelling companion.

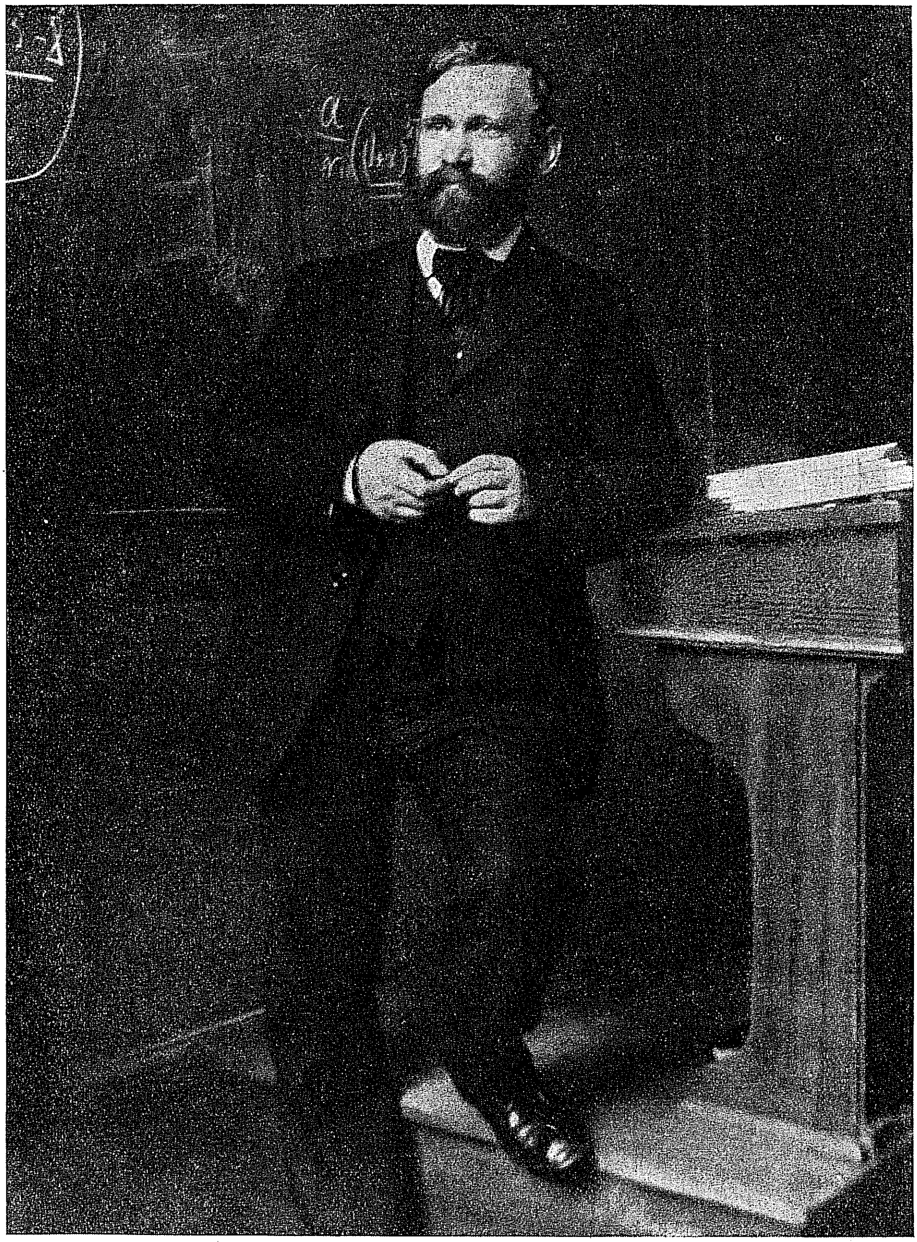
Erlang was a member of "Matematisk Forening" (Mathematicians' Association) and assisted regularly at its meetings. There, he became acquainted with the notable mathematician *J. L. W. V. Jensen*, Ph. D., then chief engineer and head of the technical department of the Copenhagen Telephone Company; as a result of this acquaintanceship, Erlang was introduced to the then managing director of the Company, *Fr. Johannsen*, D. Sc.

The merit of having introduced the theory of probabilities into telephony is due to Dr. Johannsen as he had published, in 1907 and 1908, two essays under the respective titles of "Waiting Times and Number of Calls"[1]) and "Busy"[2]), the former dealing with the delay problem in connexion with incoming calls in manual telephone exchange switchboards, and the latter being an investigation as to how often subscribers with one or more lines each are reported "busy". In both papers, the problems were coped with by means of the theory of probabilities; true, the method was not mathematically exact, but the results obtained were sufficiently correct to be serviceable for practical purposes.

Dr. Johannsen, who had a remarkable talent for selecting assistants of the right sort, suggested to Erlang that he should take these problems up for mathematical treatment. For some time Dr. Johannsen had been

[1]) Reprinted in "Telephone Management in Large Cities", Post Office Electrical Engineers Journal, London, Oct. 1910 and Jan. 1911.

[2]) Reprinted in "The Development of Telephonic Communication in Copenhagen 1881-1931", Ingeniørvidenskabelige Skrifter A, No. 32, Copenhagen, 1932, p. 150.



A. K. Erlang, 32 years old.

nursing the idea of establishing a physico-technical laboratory for scientific research work, as the development and importance of the Copenhagen Telephone Company seemed to justify such a step, and so, in 1908, the Company engaged Erlang as scientific collaborator and head of its laboratory. In this position, which gave him an opportunity to develop and utilize his great gifts and considerable knowledge, he worked for the rest of his life.

With enthusiasm and diligence, Erlang immediately set to work at applying the theory of probabilities to problems of telephone traffic, the domain that was to make his name widely known. As early as in 1909 he published his first work on this subject, "The Theory of Probabilities and Telephone Conversations", in which he proved that telephone calls distributed at random follow the Poisson law of distribution, and gave the exact solution of the delay problem stated in Dr. Johannsen's essay of 1907, in the special case of only one operator being available to handle the calls.

Erlang also cooperated with Dr. J. L. W. V. Jensen's successor-to-be as chief engineer to the Company, *P. V. Christensen*, who in 1913 published his paper on "The Number of Selectors in Automatic Telephone Exchanges"[1]) in which he, as the first, treated these problems by means of the theory of probabilities. Erlang contributed to this paper by preparing the tables stating the probability of loss.

In 1917 Erlang published his most important work, "Solution of some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges", containing his formulae for loss and waiting time which he had developed on the basis of the principle of statistical equilibrium; these now well-known formulae are of fundamental importance to the theory of telephone traffic.

In the course of the years, Erlang published several other valuable works on the theory of telephone traffic and some smaller works with reference to other mathematical domains, especially the calculation of tables of logarithms and other numerical tables. All these shall not be enumerated here, however, as a survey of Erlang's works is given on pp. 101-130.

Nearly all his works have first been published in Danish in the form of articles in various journals, but the most important of them have later been translated into one or more foreign languages such as English, French, and German, and printed in foreign journals. Erlang had a decided propensity to concise speech, and he wrote his essays in a very brief style, too; in fact, his conciseness was so ingrained that he even published many

[1]) Published, in "Elektroteknikeren", 1913, p. 207, also in Elektrotechnische Zeitschrift, 1913, p. 1314 and in Post Office Electrical Engineers Journal, 1914, p. 271.

of his results without giving the proofs. The concise style and the omission of the proofs serve, to some extent, to complicate the study of his original works for readers who are not specialists in the relevant domains.

However, Erlang's works on the theory of telephone traffic soon won recognition and understanding not only in the Scandinavian countries, but also in other countries. Thus, a few years after its appearance his formula for the probability of loss was accepted by the British Post Office as basis for calculations respecting circuit facilities. It may be mentioned as an example of the interest taken in his works that two researchers of the subject taught themselves Danish in order to be able to read Erlang's papers in the original language, namely Dr. *A. E. Vaulot*, a Frenchman who has translated some of Erlang's works into French, and Dr. *Thornton C. Fry* of the Bell Telephone Laboratories, U. S. A.

As the leader of the laboratory of the Telephone Company, Erlang had opportunity to grapple with numerous and varied physico-technical problems, one of his first being the measuring of stray currents. Prof. *Absalon Larsen* and *S. A. Faber* had previously laid the foundation by investigating the distribution of stray currents and their damaging influence upon the lead sheaths of telephone cables, but it fell to Erlang's lot to systematize the practical procedure. At first Erlang had no laboratory staff to assist him; he had to carry out all measurements of stray currents in person, and so he could be seen frequently in the streets of Copenhagen followed by a workman carrying a ladder, which was used for the purpose of climbing down into the manholes.

Erlang published some works dealing with various problems pertaining to the physico-technical side of telephony that were of considerable importance at the time of their appearance. For the purpose of measuring alternating currents, for instance, he constructed a measuring apparatus — the so-called "Erlang's Complex Compensator" — which represented a considerable improvement, compared with similar types of measuring instruments of earlier date. He did not publish many works of this kind, however, as most of his laboratory tasks consisted in the solution of concrete problems in connexion with loaded cables, transmission schemes, etcetera. A great part of these practical results of his would undoubtedly have been of interest to many, but Erlang used to think that he could not afford the time it would be necessary to spend on preparing such works for publication.

The engineers of the Telephone Company could always rely on Erlang when they needed his help in questions of physical or mathematical nature; a frequent visitor to the public libraries, he had an amazing knack of procuring literature to cover any topical matter. Erlang had his own, almost Socratic, manner of answering questions put to him: as a rule, he would hesitate to give a solution of the question asked, directly, preferring instead to enter into a sometimes lengthy discussion that would elucidate the subject from any conceivable point of view; in this manner he forced the inquirer into thinking the matter over on his own, thereby perhaps finding his way to solve the problem independently. After such a discussion with Erlang one always felt enriched far beyond the scope of its original subject.

Erlang remained single all his life. There was a time, however, when he was very much interested in a pretty young girl; but she married one of Erlang's fellow collegians, and it took him a long time to live down his disappointment. He was fond of children, and they liked him; he enjoyed to chat with children, and sometimes even taught them to play chess.

Erlang devoted all his energy to his scientific studies and his work at the laboratory of the Telephone Company. He would often work far into the night in his study at home, and when he eventually retired to bed, he armed himself with four or five books to choose between; then, he would open one of the books and begin to read, usually falling asleep and forgetting about his book and the electric light a few minutes later. When his sister discovered that the light was burning to no purpose at all, she sometimes tip-toed into his room and turned it off; it never failed, on these occasions, that he immediately woke up and exclaimed in a voice which he tried hard to make severe and offended, "I am working!" — whereupon his sister hurriedly withdrew, of course.

Erlang collected a library — a remarkably large one, at that, for a private person — chiefly comprising works about mathematics, physics, and astronomy. His knowledge of these subjects was great and comprehensive, and yet he was interested in a good many other subjects as well, such as philosophy, history, and poetry; he had a special liking for *Pascal* and his production.

Besides being a member of "Matematisk Forening", the meetings of which he attended regularly, Erlang was an associate of the British Institution of Electrical Engineers.

Erlang had a noteworthy and original personality. He was a sincere Christian in a sympathetic way, at the same time being full of humour and satirical wit; outwardly, his heavy red full beard and his manner of dressing lent a certain artistic touch to his characteristic appearance. Extremely modest and unobtrusive of demeanour, he preferred the peaceful atmosphere of his study to social gatherings and festivities; he never touched alcoholic liquors nor smoked tobacco. His mode of life has been expressed very appropriately in the words which Prof. Nybølle put as the motto of his obituary[1]) of Erlang, his intimate friend: the well-known

[1]) Published in "Matematisk Tidsskrift" B, 1929, p. 32.

quotation which we have prefixed to this biography, too. The recognition and esteem that fell to Erlang's lot, especially towards the end of his life, quite naturally made him happy, though.

Erlang was a beneficent man; living frugally, he could afford to help others, which he did to an even very great extent. His youngest sister, Miss *Ingeborg Erlang*, with whom he shared apartments for many years right up to his death, had founded a home for feeble-minded women with her own money, and in the course of the years Erlang donated a considerable part of his income to this home. Needy people often turned up at the laboratory to apply to Erlang for help, and he would invariably help them with ready money in as inconspicuous a manner as possible.

For a period of nearly 20 years Erlang had served the Copenhagen Telephone Company without a single day's absence on account of illness. In January, 1929, he felt ill, however, and it turned out that he was suffering from an abdominal disease that would necessitate an operation. Before he was put in hospital, he took leave of all his colleagues at the Company, telling them that he was going to stay in hospital for a short time only; but when the tidings of his death a few days later reached the Telephone House, his friends realized that he must have known his time was drawing near. His religious conviction and his philosophical mode of thought had bestowed upon him the serenity with which he went to meet his death.

Agner Krarup Erlang died on Sunday, the 3rd February, 1929, only 51 years old.

* * *

In the autumn of 1943, the editors of the Swedish journal "Tekniska Meddelanden från Kungl. Telegrafstyrelsen" invited — in connexion with an essay[1]) by *Conny Palm*, Sc. D. — interested parties to enter for a prize competition to be held for the purpose of finding a suitable name for the natural unit of telephone traffic. This unit had not hitherto had any particular name, for which reason such a vague denomination as "Traffic Unit" (and in other languages, correspondingly, "Trafikenhed", "Verkehrseinheit", etcetera) had been used, which might easily give occasion to confusion with other units of telephone traffic being employed in practice, for instance: *Sm*, "Speech minutes"; *E. B. H. C.* "Equated Busy Hour Calls".

In Danish quarters this gave rise to the idea of identifying A. K. Erlang's name with the natural unit of traffic. Accordingly, Mr. *P. V. Christensen*, Chief Engineer to the Copenhagen Telephone Company; Mr. *N. E. Holmblad*, Chief Engineer to the Danish Post & Telegraph Office; Prof.

[1]) *Conny Palm:* Samtalsminut eller Trafikenhet, T. M. f. K. T., 1943, p. 133.

J. Oskar Nielsen, and Prof. *J. Rybner*, both of the Royal Danish College of Engineering, sent the following note to the editors of "Tekn. Medd. fr. Kungl. Telegrafstyrelsen":

"In connexion with Dr. Conny Palm's essay: "Speech Minute or Traffic Unit" in "Tekniska Meddelanden", 1943, nos. 7-9, and the editors' invitation to enter for a prize competition with the object of finding a new name for the natural unit of telephone traffic, we wish to express our sympathy with the idea of introducing a suitable name that is well adapted to be generally approved for international application.

"We would like to connect the said unit of traffic with the name of the prominent researcher of the theory of telephone traffic, Magister *A. K. Erlang* (1878-1929), whose pioneer works in this field are universally known and appreciated. It is sufficient to mention the formula, published by Erlang in 1917, for the probability of loss in the case of a simple group of circuits, the so-called "B-formula", which must be regarded as the first mathematically exact solution of a problem of barred access; in this formula enters, besides the number of circuits (x), the intensity of traffic (y) as expressed in terms of the said natural unit of traffic.

"We beg to suggest, therefore, that the name of "*Erlang*" be introduced as the denomination of the natural unit of traffic.

"The suggested denomination is in analogy with well-known unit denominations such as Ohm, Ampère, Gauss, Maxwell, Örsted, etcetera.

"It would give us great pleasure if the adjudication committee would support our suggestion by accepting the name of "Erlang" as the denomination of the unit of traffic, thereby contributing to the introduction of this name into telephony."

The result of the competition was that the committee accepted the name of "Erlang" which, by the way, had also been suggested by 8 out of the 26 Swedish competitors. The adjudication committee gave the following three reasons for its decision:

1) "Erlang" brings the name of the founder of modern telephone traffic research to memory.

2) "Erlang" seems phonetically attractive, which is of consequence to a possible international acceptance of the name.

3) "Erlang" is not formed by abbreviation of any term in any specific language, which means the absence of any risk of metamorphosis by translation into any other language.

In consequence of this decision the name of "Erlang" has been used in the Scandinavian countries to denominate the unit of traffic since the beginning of 1944; World War II made it impossible, however, to seek international acceptance at that time.

When, after the termination of the war, the scientific intercourse of the countries was resumed, the Swedish Telephone Administration suggested to "Le comité consultatif international des communications téléphoniques à grande distance" (C. C. I. F.) that the name of "Erlang" be internationally accepted as denominating the traffic unit.

The proposal came on for trial at the plenary meeting of C. C. I. F. on the 28th October, 1946, at Montreux; Mr. N. E. Holmblad, the leader of the Danish delegation, made some remarks on this occassion, by way of commenting the proposal, upon Erlang's achievements within the theory of telephone traffic; whereupon the proposal was carried unanimously by the assembly.

Thus, "Erlang" is henceforth the international unit of telephone traffic, and in the proceedings of the C. C. I. F. the following definition is recorded:

"*The Handling of the Traffic Passing Through a Circuit or Group of Circuits.*"

"For a group of circuits (or connecting devices), the average intensity of traffic during a period T equals the total occupancy divided by T.

"The unit of traffic intensity as defined above is called "*erlang*".

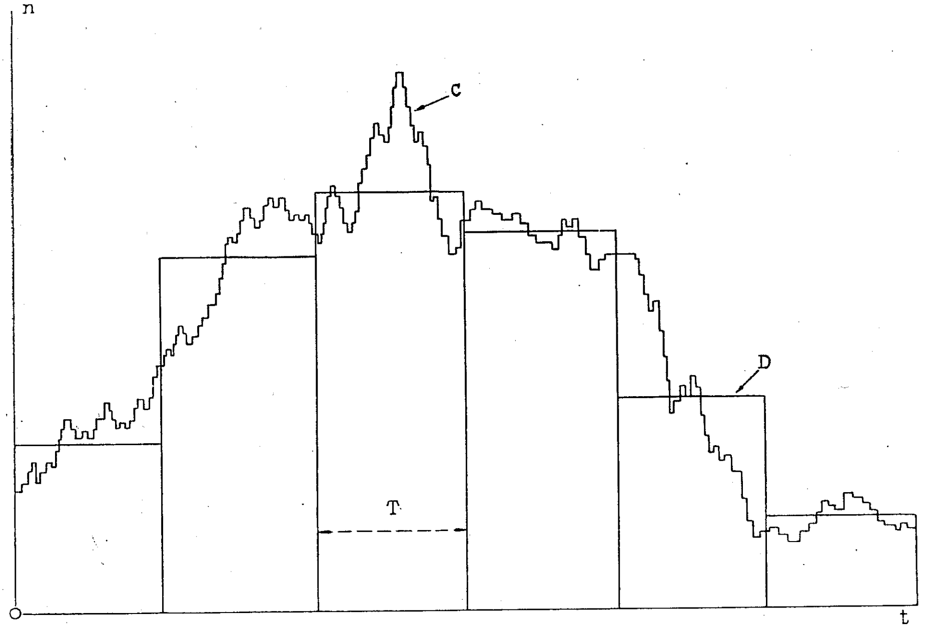
"Explanatory note:

"The number of connecting devices or circuits that are occupied at any particular point of time can be ascertained by means of automatic devices designed for the purpose. Let the time t be marked out as abscissa, and let the ascertained number of simultaneously occupied circuits n be marked out as ordinate; a curve can then be obtained, similar to the curve C shown in the affixed figure. The mathematical expression for the average intensity of traffic I_m is given by the formula,

$$I_m = \frac{\int_{t_0}^{t_0+T} n \cdot dt}{T},$$

where t_0 denotes the point of time when the period T begins.

"Notwithstanding the fact that the average intensity of traffic is thus expressed by a quantity of no physical dimension, the Committee considers it expedient to suffix the word "erlang" to the said quantity, in order to characterize the same, just as the terms of "neper" or "decibel" are employed to express values of attenuation. An average value of traffic intensity is usually involved in calculations of the number of connecting devices, so that the step-like diagram D (shown in the figure) is used instead of the curve C. Each of the rectangles in this diagram has the same area as that bounded by the corresponding segment of the



curve. The height of each rectangle is equal to the quantity, expressed in "erlang", that characterizes the average intensity of traffic during the period represented by the base of the rectangle; if total occupancy of all connecting devices never occurs during the period T, the said quantity in terms of "erlang" also expresses the amount of traffic to be handled.

"The name of "erlang" is accepted in recognition of the important investigations of the applicability of the theory of probabilities to telephonic problems that were carried out by the Danish scientist *Erlang*."

AN ELUCIDATION OF ERLANG'S STATISTICAL WORKS THROUGH THE THEORY OF STOCHASTIC PROCESSES

By ARNE JENSEN

Introduction.

A. K. Erlang developed some theories belonging under the science of statistics which have been of special importance to the solution of problems within the field of telephone technics, where their applicability soon received due recognition. In fact, these theories broke new ground; but they are not very well known outside the field of telephony, owing to the fact that Erlang's published works appear only in the form of solutions of a few special problems where the principles, upon which his solutions are based, do not stand out clearly.

By emphasizing the method of procedure and the generalities in the following, I have attempted to throw light on a number of particular solutions which have hitherto been but little known on account of the form of representation. Furthermore, in order to lay stress on the fact that the problems treated by Erlang are in reality statistic problems with a wider scope of applicability, I have tried to break away, to some extent, from the terminology of telephony.

Erlang utilized a very useful domain of the so-called stochastic processes, the complete theory of which was not given, as it happens, until much later. The application of the theory of these stochastic processes has, however, resulted in an expansion of Erlang's work and thereby made it possible to expound the underlying principles more clearly.

A simple expansion of the basic principles of Erlang's holding time distributions will lead to a system of distributions that is of interest to the general theory of distributions.

The notation now commonly used has been adopted in the following, instead of Erlang's own notation.

Erlang's principal statistical works, translated into English, are reprinted on pp. 131—215.

On pp. 101—108, Mr. *E. Brockmeyer*, M. Sc., gives a brief survey of the contents of Erlang's various statistical articles.

I remember gratefully my talks with the late Professor *H. Cl. Nybølle*, one of Erlang's few near friends, with whom Erlang had many debates about the problems treated in this book. These talks have been a great help to me, as Erlang left but few notes apart from his published articles.

Finally I want to thank Professor *A. Hj. Hald*, Ph. D., *G. Rasch*, Ph. D., Lecturer in the University of Copenhagen, Miss *Vibeke Borchsenius*, M. Sc., and Mr. *E. Brockmeyer*, M. Sc., for detailed discussions of the problems dealt with in the following.

1. Laws of Distribution.

In his works, Erlang mentions several laws of distribution for the time during which a connecting device, an operator, a position, or the like, is occupied with a call, or is otherwise occupied. They can all be derived from the following simple assumption:

The termination of a call in progress depends upon the previous occurrence of a certain number of events, the nature of these events being specified in each case. The probability that a discrete event will occur within a certain time interval is asymptotically proportional to the length of the time interval, with a factor of proportionality λ that is independent of time. (1.1)

Hence it follows that the probability for the number of events ν occurring within the time interval t is Poisson's law of distribution with the mean λt. This distribution can be written

$$p(\nu) = \frac{(\lambda t)^\nu}{\nu!} e^{-\lambda t} \tag{1.2}$$

where $p(\nu)$ is the probability that ν events will occur in the time interval t. If the termination of the call coincides with the occurrence of the fth event, the probability $P(> t)$ that the call will not yet have come to an end at the expiration of the time t after the initiation of the call, will be equal to the probability that at most $f - 1$ events will have occurred in the time interval t. If the above mentioned factor of proportionality here is $f\lambda$, it follows that

$$\begin{aligned} P(> t) &= \sum_{\nu=0}^{f-1} p(\nu) \\ &= e^{-f\lambda t} \sum_{\nu=0}^{f-1} \frac{(f\lambda t)^\nu}{\nu!}; \end{aligned} \tag{1.3}$$

this can be put in the form

$$P(> t) = \int_{f\lambda t}^{\infty} \frac{y^{f-1}}{(f-1)!} e^{-y}\, dy, \qquad t \geqq 0, \tag{1.4}$$

which can be shown by partial integration.

The probability $p(t)dt$ that the duration of the call, or rather, the holding time, will be at least t and at most $t + dt$ is, then,

$$p(t)dt = \frac{(f\lambda t)^{f-1}}{(f-1)!} e^{-f\lambda t} f \lambda \, dt, \qquad t \geqq 0, \tag{1.5}$$

which is the so-called Type III of Pearson's system of distributions, also known as the χ^2-*distribution*, with $2f$ *degrees of freedom*.

The mean value M of the distribution is

$$M = \int_0^\infty P(>t)\, dt = \frac{1}{\lambda}, \tag{1.6}$$

that is to say, the distribution (1.5) has the same mean value for all values of f. Fig. 1.1 shows a graphical representation of the distribution for different values of f, and $\lambda = 1$.

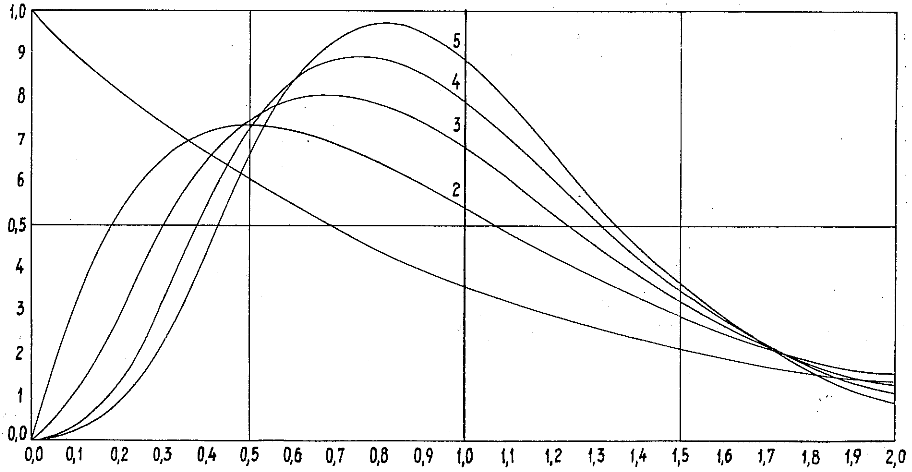


Fig. 1.1.
The distribution (1.5) for different values of f.

For $f = 1$ we get the distribution that is most frequently used in telephone engineering:

$$p(t)\, dt = e^{-\lambda t} \lambda dt. \tag{1.7}$$

Using the Incomplete Gamma Function and its properties we have for $f \to \infty$ in (1.4) that

$$\lim_{f \to \infty} P(> t) = \begin{cases} 0 & t \geqq \frac{1}{\lambda} \\ 1 & t < \frac{1}{\lambda} \end{cases}, \tag{1.8}$$

which is the sum function of another law of distribution frequently used in telephone engineering:

$$p(t) = \begin{cases} 1 & t = \frac{1}{\lambda} \\ 0 & t \neq \frac{1}{\lambda} \end{cases}. \tag{1.9}$$

A more general system of distributions, which, *inter alia*, comprises the distributions employed by Erlang (1.5), can be derived on the basis of the following consideration:

The termination of a call in progress depends upon the previous occurrence, in a prescribed order, of a certain number of events f. The probability that a discrete event will occur within a certain time interval is asymptotically proportional to the length of the time interval, with a factor of proportionality that, for the νth event, is λ_ν. The termination of the call will coincide with the occurrence of the fth event after the initiation of the call. It is temporarily assumed that $\lambda_\nu \neq \lambda_\mu$ for $\nu \neq \mu$. (1.10)

Let t_0 be the time at which the call was originated, and let $p(t, \nu)$ be the probability that *exactly* ν events have occurred before the time t; we have then, using (1.7), that the probability that the first event has not yet occurred at the time t is

$$p(t, 0) = e^{-\lambda_1 (t - t_0)}, \qquad t \geqq t_0. \tag{1.11}$$

The probability that the first event, but only the first, will have occurred before the time t is, then,

$$p(t, 1) = \int_{t_0}^{t} p(t_1, 0)\, \lambda_1\, e^{-\lambda_2 (t - t_1)}\, dt_1 \tag{1.12}$$

$$= \lambda_1 \left[\frac{e^{-\lambda_1 (t - t_0)}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 (t - t_0)}}{\lambda_1 - \lambda_2} \right], \tag{1.12}$$

and the probability that *exactly the μth* of the f events, but no more, will have occurred before the time t is

$$p(t, \mu) = \int_{t_0}^{t} p(t_1, \mu - 1)\, \lambda_\mu\, e^{-\lambda_{\mu+1}(t-t_1)}\, dt_1, \qquad \begin{matrix} \mu \leqq f - 1 \\ t \geqq t_0 \end{matrix}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_\mu \sum_{\nu=1}^{\mu+1} \frac{e^{-\lambda_\nu (t-t_0)}}{\prod\limits_{\substack{i=1 \\ i \neq \nu}}^{\mu+1} (\lambda_i - \lambda_\nu)} \tag{1.13}$$

Hence it follows that the probability that the holding time will exceed the time t is

$$P(>t, f) = \sum_{\mu=0}^{f-1} p(t, \mu)$$

$$= \lambda_1 \lambda_2 \cdots \lambda_f \sum_{\nu=1}^{f} \frac{e^{-\lambda_\nu (t-t_0)}}{\lambda_\nu \prod\limits_{\substack{i=0 \\ i \neq \nu}}^{f} (\lambda_i - \lambda_\nu)} \tag{1.14}$$

and that the probability that the holding time will come to an end at a time between t and $t + dt$ is

$$p(t, f)\, dt = \lambda_1 \cdots \lambda_f \sum_{\nu=1}^{f} \frac{e^{-\lambda_\nu (t-t_0)}}{\prod\limits_{\substack{i=1 \\ i \neq \nu}}^{f} (\lambda_i - \lambda_\nu)}\, dt, \qquad t \geqq t_0.\ ^{1)} \tag{1.15}$$

The law of distribution is the limiting value of this expression when some of the constants λ_i are equal; thus when all $\lambda_i = \lambda$, we obtain (1.5). A slight alteration of the assumption (1.10) will lead to a system of discontinuous distributions. The discontinuous distribution corresponding to (1.15) is obtained by letting t be given instead of f:

$$p(f, t) = \lambda_1 \cdots \lambda_{f-1} \sum_{\nu=1}^{f} \frac{e^{-\lambda_\nu (t-t_0)}}{\prod\limits_{\substack{i=1 \\ i \neq \nu}}^{f} (\lambda_i - \lambda_\nu)}, \qquad t \geqq t_0.\ ^{2)} \tag{1.16}$$

The system of distributions indicated by (1.15) and (1.16) is rather comprehensive.

For $\lambda_\nu = \alpha + \beta \cdot \nu$ we obtain, *e. g.*, the following distributions:

[1]) This distribution was first published by *C. Palm*, (1946).

[2]) This distribution was first published by *O. Lundberg*, (1940).

$$p(t, f)\, dt = \binom{f-1+\frac{\alpha}{\beta}}{f-1} e^{-(\alpha+\beta)(t-t_0)} (1 - e^{-\beta(t-t_0)})^{f-1} (\alpha + f\beta)\, dt \quad (1.17)$$

$$\alpha + \beta\nu > 0 \qquad \nu = 1, \cdots, f,$$
$$t \geqq t_0,$$

and the corresponding distribution

$$p(t, f) = \binom{f-1+\frac{\alpha}{\beta}}{f-1} e^{-(\alpha+\beta)(t-t_0)} (1 - e^{-\beta(t-t_0)})^{f-1}, \quad (1.18)$$

$$\alpha + \beta\nu > 0, \quad \nu = 1, 2, \cdots,$$
$$f = 1, 2, \cdots.$$

which, by a simple transformation of the time, contains the *Pearson distributions* and the so-called *Pólya distribution.*

2. *On Discontinuous Stochastic Processes.*

The distribution functions developed by Erlang may be interpreted as limiting functions of certain discontinuous stochastic processes, the properties of which shall be elucidated in the following through an examination of a special case, the *Poisson process.*

We want to find the law of distribution of a number of events i occurring during a time interval t after a point of time T, on the assumption that the probability that any one event will occur within the time interval t is proportional to the length of t with a certain factor of proportionality and independent of the point of time T. The event may be, for instance, a pedestrian's passing of a certain spot in a street, or the arrival of a telephone call in a certain group of switches in a telephone exchange. Such a law of distribution is called a stochastic, or random, process.

It may be useful to add a brief explanation since the last mentioned example will be used frequently in the following. In a telephone system, a group of subscribers has access to a certain number of connecting devices such as selectors in the exchange, and conductors in cables between exchanges, so that the calls originated by the said subscribers must pass through the said connecting devices. Any such connecting device which happens to be carrying a conversation is blocked for new calls for the duration of the conversation already going on. A calling subscriber is therefore referred to other connecting devices; if no others are available, he either loses his call, or he will have to wait for a free switch, depending on the exchange system used. In the following the term "switch" not only denotes a pair of conductors in a cable, but also a selector or any other such connecting device. A switch carrying a call or conversation is said to be "occupied" by a call or conversation.

The factor of proportionality mentioned above is called λ; the probability that the occurrence of a total of i events may be observed during the time t is $P(i, t)$; and (t_1, t_2) denotes the time interval from t_1 to t_2.

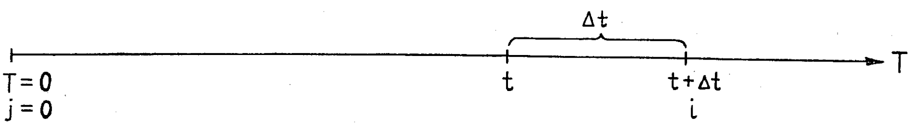


Fig. 2.1.

It is assumed that no event ($j = 0$, see Fig. 2.1) has yet occurred at the time origin $T = 0$.

In order to determine $P(i, t)$, we derive $P(i, t + \Delta t)$ on the basis of the events occurring during the two intervals $(0, t)$ and $(t, t + \Delta t)$. By a subsequent limit passage we obtain differential equations from which $P(i, t)$ can be determined. As the events occurring in the two time intervals under consideration are mutually independent, the probability that no event will occur before the time $t + \Delta t$ is the product of the probability for no events in the interval $(0, t)$ and the probability for no events in the interval $(t, t + \Delta t)$. The probability for no events in the interval $(0, t)$ is $P(0, t)$; the probability that an event will occur in the interval $(t, t + \Delta t)$ is asymptotically equal to $\lambda \Delta t$. Hence it follows that the probability for no event in the said interval is asymptotically equal to $1 - \lambda \Delta t$, that is to say,

$$P(0, t + \Delta t) = P(0, t)(1 - \lambda \Delta t) + o(\Delta t) \tag{2.1}$$

where

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$$

Now, i events can occur in the interval $(0, t + \Delta t)$ in various ways. All the i events may have occurred within the time interval $(0, t)$, and therefore no events in the interval $(t, t + \Delta t)$; or $i - 1$ events may have occurred in the former interval, and 1 in the latter; or $i - 2$ in the former, and 2 in the latter; or $\cdots$; or 0 in the former, and i in the latter interval. These different possibilities are mutually exclusive. The probability $P(i, t + \Delta t)$ is therefore the sum of the probabilities that i will be distributed in the above mentioned ways. The probability of any one of these is the product of the probability of the given number of events in the interval $(0, t)$ and the probability of the given number of events in the interval $(t, t+\Delta t)$; therefore,

$$P(i, t+\Delta t) = P(i, t)(1-\lambda \Delta t) + P(i-1, t)\lambda \Delta t + o(\Delta t). \tag{2.2}$$

(2.1) and (2.2) can be written in the form

$$\frac{P(0, t+\Delta t)-P(0, t)}{\Delta t} = -\lambda P(0, t) + \frac{o(\Delta t)}{\Delta t} \tag{2.3}$$

$$\frac{P(i, t+\Delta t)-P(i, t)}{\Delta t} = +\lambda P(i-1, t) - \lambda P(i, t) + \frac{o(\Delta t)}{\Delta t}, \quad i > 0, \tag{2.4}$$

but

$$\lim_{\Delta t \to 0} \frac{P(i, t+\Delta t)-P(i, t)}{\Delta t} = P'_t(i, t), \quad i \geqq 0, \tag{2.5}$$

so that (2.3) and (2.4) for $\Delta t \to 0$ lead to

$$P'_t(0, t) = -\lambda P(0, t) \tag{2.6}$$

$$P'_t(i, t) = \lambda P(i-1, t) - \lambda P(i, t), \quad i > 0. \tag{2.7}$$

As it is assumed that the number of events is zero at the time origin $T = 0$, we have that the initial value $(j, T) = (0, 0)$ and, therefore,

$$P(i, 0) = \begin{cases} 1, & i = 0 \\ 0, & i > 0. \end{cases} \tag{2.8}$$

The differential equations (2.6) and (2.7) have one and only one solution satisfying (2.8), *viz.*,

$$P(i, t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}. \tag{2.9}$$

Since this stochastic process for a fixed value of t represents the *Poisson law of distribution*, it is called *Poisson's Process*. Its mean value λt is monotonic increasing, and it is continuously dependent on the parameter t. A stochastic process may be characterized, for instance, as a distribution whose parameters are dependent on the same variable, here the time t.

When the initial value is $(j, 0)$ instead of $(0, 0)$, (2.8) is altered into

$$P(i, 0) = \begin{cases} 1, & i = j, \\ 0, & i > j, \end{cases} \quad i \geqq j. \tag{2.10}$$

(2.6) and (2.7) have one and only one solution satisfying (2.10), *viz.*,

$$P(i, t \mid j, 0) = \frac{(\lambda t)^{i-j}}{(i-j)!} e^{-\lambda t}, \quad i \geqq j, \tag{2.11}$$

where the notation $P(i, t \mid j, 0)$ expresses that the stochastic process is a conditional probability, the condition being that the initial value must be $(j, 0)$.

When the initial value is (j, t_0) instead of $(j, 0)$, (2.10) is altered into

$$P(i, t_0 \mid j, t_0) = \begin{cases} 1, & i = j, \\ 0, & i > j, \end{cases} \quad i \geqq j. \tag{2.12}$$

(2.6) and (2.7) have one and only one solution satisfying (2.12), *viz.*,

$$P(i, t \mid j, t_0) = \frac{(\lambda(t - t_0))^{i-j}}{(i-j)!} e^{-\lambda(t-t_0)}, \quad \begin{matrix} i \geqq j \\ t \geqq t_0. \end{matrix} \tag{2.13}$$

This general expression of the Poisson process serves to illustrate an ordinary stochastic process; it indicates the probability for a state i at a time $T = t$ $(t \geqq t_0)$, assuming that the state j was prevailing at the time $T = t_0$.

The Poisson process (2.13) satisfies the inequality

$$0 \leqq P(i, t \mid j, t_0) \leqq 1, \quad \begin{matrix} i \geqq j \\ t \geqq t_0, \end{matrix} \tag{2.14}$$

and the equation

$$\sum_{i=j}^{\infty} P(i, t \mid j, t_0) = 1, \quad t \geqq t_0, \tag{2.15}$$

where the process of summation includes all possible values of i.

The Poisson process is stochastically definite (it is said to belong to the so-called Markoff chains), as it satisfies the conditional equation

$$P(i, t \mid j, t_0) = \sum_{j'} P(i, t \mid j', t') \, P(j', t' \mid j, t_0), \quad t_0 \leqq t' \leqq t, \tag{2.16}$$

where the process of summation includes all possible values of j'. The relation (2.16) is called *Chapman-Kolmogoroff's equation.*

The Poisson process satisfies (2.16) because

$$P(i, t \mid j, t_0) = \sum_{j'=j}^{i} \frac{(\lambda(t - t'))^{i-j'}}{(i-j')!} \frac{(\lambda(t' - t_0))^{j'-j}}{(j'-j)!} e^{-\lambda(t-t')} e^{-\lambda(t'-t_0)} \tag{2.17}$$

$$= e^{-\lambda(t-t_0)} \frac{1}{(i-j)!} \frac{(\lambda(t - t_0))^{i-j}}{(i-j)!},$$

which is (2.13).

It follows from (2.16) that the probability depending on the two conditions (j', t') and (j, t_0) satisfies the following equation:

$$P(i, t \mid j', t' \mid j, t_0) = P(i, t \mid j', t'), \quad t_0 < t' < t, \tag{2.18}$$

which is an expansion of

$$P(i, t \mid j', t') = P(i, t). \tag{2.19}$$

The relation (2.19) is the general condition for stochastic independence of events. The relation (2.18) expresses that only the latest information about the state, (j', t'), will influence the conditional probability relative to previously occurred events, for instance (j', t') and (j, t_0).[1])

We shall briefly mention some other concepts.

A stochastic process is said to be *homogeneous with respect to time* when the conditional probability is exclusively dependent on the length of the time interval under consideration, and not on the points of time at which it begins or ends, such as when

$$P(i, t \mid j, t_0) = P(i, t - t_0 \mid j, 0) \tag{2.20}$$

A stochastic process is said to be *homogeneous with respect to space* when the conditional probability is exclusively dependent on the changes in the state of events from the initial time t_0 to the point of time t under consideration, and not on the initial state proper or the final state proper, such as when

$$P(i, t \mid j, t_0) = P(i - j, t \mid 0, t_0) \tag{2.21}$$

When (2.20) and (2.21) are satisfied, the process is called *homogeneous.*

As a stochastic process must satisfy

$$P(i, t_0 \mid j, t_0) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{2.22}$$

we have, on certain conditions[2]), that

$$\sum_{i=0}^{\infty} P(i, t \mid j, t_0) \equiv 1. \tag{2.23}$$

The conditions are satisfied in the case of the stochastic processes to be treated in the following, if

$$a_\nu + \beta_{\nu j} \leqq k_1, \qquad \nu = 1, 2, \cdots \tag{2.24}$$

or

$$a_\nu + \beta_{\nu j} \leqq k_2 \; \nu, \qquad \nu = 1, 2, \cdots, \tag{2.25}$$

[1]) For further information, see *A. Kolmogoroff*, 1931 and 1933, *R. v. Mises*, 1931, *B. Hostinsky*, 1931.

[2]) For further information, see *N. Arley and V. Borchsenius*, 1945, p. 286.

where $\alpha_\nu \Delta t$ and $\beta_{\nu f} \Delta t$ are asymptotic probabilities that a change from ν to $\nu - 1$, or from ν to $\nu + f$, respectively, will take place during the time interval $(t, t + \Delta t)$.

It will appear from (2.13), (2.11), and (2.2) that the *Poisson process is homogeneous* and satisfies (2.24), as $0 = \alpha_\nu$ and $\lambda = \beta_{\nu 1}$; hence it follows that also (2.23) is satisfied.

3. The Poisson Distribution.

The process described in the preceding chapter is a pure "propagation process" where the number of events (hereinafter called individuals) is a never decreasing function of time, whereas, in all the processes considered by Erlang, the increments may be negative as well as positive (hereinafter called departures and arrivals, respectively). Thus, while the probability for a certain number of individuals i always varies as the time varies in the "propagation process" (2.13), there is a possibility that this will not be the case in the processes to be considered below, on account of the departures.

The number of individuals in the "propagation process" (2.13) means the number of telephone calls or pedestrians *passing* a certain point in the course of the time t; but in Erlang's corresponding process, the number of individuals denotes the number of conversations in progress in an unlimited group of switches, or the number of pedestrians finding themselves in a given stretch of street, at the time t. An arrival at the time t means, in these cases, the start of a new conversation, or another pedestrian's entering the street, at the time t.

The Poisson distribution considered by Erlang is based on the following two assumptions, the first of which is the same as that used in the case of the "propagation process":

The probability that an arrival will occur is asymptotically proportional to the length of the time interval under consideration and independent of the time origin under consideration. (3.1)

The probability that a departure will occur, i. e. that a certain individual will cease to be present, is asymptotically proportional to the length of the time interval under consideration and independent of the time origin under consideration. (3.2)

The respective factors of proportionality are called λ_a and λ_d.

The sought probability $P(i, t \mid j, t_0)$ is the probability that there will be i individuals (conversations or pedestrians) present at the time t, when j individuals were present at the time t_0 previous to t. We shall use the shorter denotation $P(i,t)$ where there is no risk of confusion; the initial value is tacitly assumed to be (j, t_0).

We will now investigate the probability that there will be i individuals at the time $t + \Delta t$ by finding all possible combinations of numbers of individuals at the time t, and of numbers of arrivals and departures during the time interval Δt, that will result in the presence of i individuals at the time $t + \Delta t$. By letting $\Delta t \to 0$ we can then obtain a system of differential equations from which $P(i,t)$ can be determined.

The investigation is similar to that of the Poisson process previously treated in that we have to consider the two cases of $i = 0$ and $i > 0$.

$\underline{i = 0.}$

The following combinations will give 0 individuals at the time $t + \Delta t$:

Number of individuals at time t:	Arrivals during interval Δt:	Departures during interval Δt:	
0	0	0	(3.3)
1	0	1	(3.4)

(Some other combinations will give the same result, such as: 2 individuals, 1 arrival, and 2 departures, &c.; but then the probabilities will be asymptotically equal to 0, as a computation would show.)

The probabilities corresponding to the above cases are

$$P(0,t) \quad (1 - \lambda_a \Delta t) \quad 1 \tag{3.5}$$

$$P(1,t) \quad (1 - \lambda_a \Delta t) \quad 1\, \lambda_d \Delta t \tag{3.6}$$

The probability for the combination (3.3) is the product of the two probabilities in (3.5), and the probability for the combination (3.4) is the product of the three probabilities in (3.6), as the probabilities for numbers of individuals, arrivals, and departures, are independent of each other. The probability for the other combinations is $o(\Delta t)$. Accordingly, $P(0, t + \Delta t)$ is the sum of the probabilities thus derived, since the combinations stated are mutually exclusive.

We have therefore

$$\begin{aligned} P(0, t + \Delta t) = {} & P(0,t)\,(1 - \lambda_a \Delta t) \\ & + P(1,t)\,(1 - \lambda_a \Delta t)\,\lambda_d \Delta t \\ & + o(\Delta t) \end{aligned} \tag{3.7}$$

$i > 0.$

The following combinations will give i individuals at the time $t + \Delta t$:

Number of individuals at time t:	Arrivals	Departures	
	during interval Δt:		
$i - 1$	1	0	(3.8)
i	0	0	(3.9)
$i + 1$	0	1	(3.10)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$$P(i-1,t) \qquad \lambda_a \Delta t \qquad (1-(i-1)\lambda_d \Delta t) \tag{3.11}$$

$$P(i,t) \qquad 1-\lambda_a \Delta t \qquad 1-i\lambda_d \Delta t \tag{3.12}$$

$$P(i+1,t) \qquad 1-\lambda_a \Delta t \qquad (i+1)\lambda_d \Delta t. \tag{3.13}$$

The probabilities for the combinations (3.8), (3.9), and (3.10), are the products of the probabilities in (3.11), (3.12), and (3.13), respectively, in analogy with the above. The probability that there will be i individuals at the time $t + \Delta t$ is the sum of the terms thus obtained, since the combinations are mutually exclusive:

$$\begin{aligned} P(i,t+\Delta t) = {} & P(i-1,t)\lambda_a \Delta t\,(1-(i-1)\lambda_d \Delta t) \\ & + P(i,t)(1-\lambda_a \Delta t)(1-i\lambda_d \Delta t) \\ & + P(i+1,t)(1-\lambda_a \Delta t)(i+1)\lambda_d \Delta t \\ & + o(\Delta t), \qquad i > 0. \end{aligned} \tag{3.14}$$

(3.7) and (3.14) may be written

$$\frac{P(0,t+\Delta t)-P(0,t)}{\Delta t} = -\lambda_a P(0,t) + \lambda_d P(1,t) + \frac{o(\Delta t)}{\Delta t} \tag{3.15}$$

and

$$\frac{P(i,t+\Delta t)-P(i,t)}{\Delta t} = \lambda_a P(i-1,t) - (\lambda_a + i\lambda_d)P(i,t) + $$

$$(i+1)\lambda_d P(i+1,t) + \frac{o(\Delta t)}{\Delta t}, \quad i > 0. \tag{3.16}$$

Using (2.5) we obtain from (3.15) and (3.16) for $\Delta t \to 0$

$$P'_t(0,t) = -\lambda_a P(0,t) + \lambda_d P(1,t) \tag{3.17}$$

$$P'_t(i,t) = \lambda_a P(i-1,t) - (\lambda_a + i\lambda_d)P(i,t) + (i+1)\lambda_d P(i+1,t) \tag{3.18}$$

$$i > 0.$$

It was assumed that j individuals were present at the time t_0; hence it follows that

$$P(i,t_0 \mid j,t_0) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{3.19}$$

which we shall call the initial condition.

Now if we may suppose that

$$\lim_{t \to \infty} P(i,t \mid j,t_0) = P(i), \quad i \geqq 0, \tag{3.20}$$

where not all $P(i) = 0$, and where the limiting distribution $P(i)$ is independent of j and t_0, then (3.20) in connexion with (3.17) and (3.18) will lead to

$$\lim_{t \to \infty} P'_t(i,t) = 0. \tag{3.21}$$

It follows from the initial condition (3.19) in connexion with (2.25) that

$$\sum_{i=0}^{\infty} P(i,t \mid j,t_0) \equiv 1 \tag{3.22}$$

because

$$\alpha_\nu = \nu\,\lambda_d \text{ and } \beta_\nu = \lambda_a \tag{3.23}$$

It follows from (3.22) and (3.20) that

$$\lim_{t \to \infty} \sum_{i=0}^{\infty} P(i,t \mid j,t_0) = \sum_{i=0}^{\infty} \lim_{t \to \infty} P(i,t \mid j,t_0) = \sum_{i=0}^{\infty} P(i) \equiv 1 \tag{3.24}$$

if $\sum_{i=0}^{N} P(i,t \mid j,t_0)$ converges uniformly to its limiting value $\sum_{i=0}^{N} P(i)$. This condition is satisfied if there exists a number M such that

$$\frac{\lambda_a}{(\nu+1)\,\lambda_d} \leqq k < 1 \quad \text{for} \quad \nu > M. \tag{3.25}$$

(3.24) will always be satisfied for the process under consideration, because (3.25) is certain to be satisfied for any set of values of λ_a and λ_d.

Thus, the limiting distribution $P(i)$ can be uniquely defined by the system of equations (3.17) and (3.18) in connexion with (3.21) and (3.24), the solution being

$$P(i) = \frac{\left(\frac{\lambda_a}{\lambda_d}\right)^i}{i!}\, e^{-\frac{\lambda_a}{\lambda_d}}, \tag{3.26}$$

which is Poisson's law, but different in form and meaning from (2.13).

Erlang has used the phrase that processes which, like (3.20), have limiting values independent of j and t_0, may enter into "*statistical equilibrium*". In the following, a process is said to enter into *statistical equilibrium* if

$$\lim_{t \to \infty} P(i,t \mid j,t_0) = P(i), \tag{3.27}$$

where not all $P(i) = 0$, and where $P(i)$ is independent of j and t_0, and if the process furthermore satisfies the necessary conditions for a distribution

$$\sum_i P(i,t \mid j,t_0) = 1 \tag{3.28}$$

$$\sum_i P(i) = 1. \tag{3.29}$$

Hence it follows, for all processes to be mentioned here, that

$$\lim_{t \to \infty} P_i'(i,t \mid j,t_0) = 0. \tag{3.30}$$

Erlang has never described "*statistical equilibrium*" as a limiting value of a stochastic process; at his time, however, the mathematicians were not particularly interested in what happened *en route* in the development of such processes, but rather in the results: the non-conditional probabilities[1]). What made an epoch was his early use of the transition probabilities and the results of statistical equilibrium, (3.21) and (3.24), in his works; he utilized, thereby, those of the stochastic processes that enter into a state of equilibrium which is independent of the initial state.

No proof respecting the conditions under which such a state of equilibrium may justly be supposed to exist has been found among Erlang's papers; nor has he ever published the general law of distribution which it would otherwise have been natural to formulate. He applied the principle of statistical equilibrium to concrete problems only, and only when he had good reason to believe that there was no risk of misusing it.

It will be noticed that only the ratio between the intensities of arrivals and departures (and not their absolute numbers) is of importance to the law of distribution respecting the state of equilibrium $P(i)$. The absolute values of the intensities are involved, on the other hand, in the general solution of (3.17), (3.18), and (3.19), where they are especially significant for small values of $(t - t_0)$.

It follows from the assumption (3.1) that the distribution for the total number of arrivals during a certain time interval follows the distribution (2.13) with $\lambda = \lambda_a$.

It follows from the assumption (3.2) that the distribution for the time

[1]) See f. inst. *L. Bachelier* (1900) and (1912).

t during which an individual is present (that is to say, the duration of a telephone conversation, or the time for which a pedestrian stays in a given stretch of street) is

$$p(t)\,dt = \lambda_d\, e^{-\lambda_d t}\, dt, \qquad (3.31)$$

the average duration of the stay (the average holding time) being $\frac{1}{\lambda_d}$, that is, the distribution (1.7) for $\lambda = \lambda_d$.

The demonstration would become more complicated for a different distribution of the staying time. This holds good not only for the stochastic processes considered here, but also for the determination of the law of distribution in case of statistical equilibrium even though this is not always dependent on the distribution of the staying time. Erlang did not always take this into consideration in his demonstrations; but he was well aware of the significance of the holding time distribution.

4. The Binomial Distribution.

In the preceding chapter dealing with the Poisson distribution, the assumptions did not contain any restrictions as to the number of individuals that might be present simultaneously. Such restrictions exist, however, in various practical investigations. Consider *e. g.* a subscriber's cable containing a certain number of circuits N; there are N subscribers connected to the cable; at most N subscribers can be engaged in conversations over the N circuits simultaneously; a subscriber engaged in one conversation cannot start a new conversation on the same circuit while the first is still going on; no new conversations can therefore be started if all N circuits are occupied. The transportation of grains of sand in a river affords another example: a grain of sand will be either moving or at rest — it must be at rest before it can start moving. In other words, a limited number of individuals is divided between 2 groups, *viz.* individuals being observed, and individuals not being observed; addition to one group equals subtraction from the other.

The assumption respecting arrivals and departures in the case of the process to be considered in this chapter may therefore be expressed as follows, (4.2) being an extended form of (3.2):

The total number of individuals is N, divided between 2 groups. (4.1)

The probability that a certain individual belonging to one group shall go over to the other group is asymptotically proportional to the length of the time interval under consideration and independent of the initial point of the time interval under consideration. (4.2)

The factor of proportionality is called λ_a by transition of individuals from the group not under observation to the group under observation, which transition constitutes an increase of the number of individuals belonging to the observed group. The factor of proportionality is called λ_d by transition from the group under observation to the group not under observation.

The probability for i individuals under observation (subscribers engaged in conversation, grains of sand in motion) at the time t, when there were j individuals at the time t_0 previous to t, can be determined by considerations similar to those on which our discussion of the Poisson distribution was based. The probability for finding i individuals under observation at the time $t + \Delta t$ is investigated by means of an enumeration of all possible combinations of numbers of individuals under observation at the time t and numbers of arrivals and departures during the short time Δt resulting in i individuals under observation at the time $t + \Delta t$. By making a limit passage, a system of differential equations for the determination of $P(i,t)$ can be obtained. The investigation is here divided into the 3 cases of $i = 0$, $0 < i < N$, and $i = N$.

$i = 0.$

The following combinations will give 0 individuals at the time $t + \Delta t$:

Number of individuals under observation at time t:	Arrivals during interval Δt:	Departures during interval Δt:	
0	0	0	(4.3)
1	0	1	(4.4)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(0,t)$	$(1 - N\lambda_a \Delta t)$	1	(4.5)
$P(1,t)$	$(1 - (N-1)\lambda_a \Delta t)$	$1\,\lambda_d \Delta t$	(4.6)

The probability for the combination (4.3) is the product of the probabilities in (4.5), and the probability for (4.4) is the product of the probabilities in (4.6), the probabilities in (4.5) and (4.6) being mutually independent. Since the combinations stated above are mutually exclusive, we have that $P(0,t + \Delta t)$ is the sum of the terms thus obtained:

$$\begin{aligned} P(0,t+\Delta t) = {} & P(0,t)\,(1 - N\lambda_a \Delta t) \\ & + P(1,t)\,(1-(N-1)\lambda_a \Delta t)\,\lambda_d \Delta t \\ & + o(\Delta t). \end{aligned} \tag{4.7}$$

$0 < i < N.$

The following combinations will give i individuals under observation at the time $t + \Delta t$:

Number of individuals under observation at time t:	Arrivals	Departures	
	during interval Δt:		
$i - 1$	1	0	(4.8)
i	0	0	(4.9)
$i + 1$	0	1	(4.10)

(and some other combinations for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$$P(i-1,t) \qquad (N-i+1)\,\lambda_a\,\Delta t \qquad 1-(i-1)\,\lambda_d\,\Delta t \tag{4.11}$$

$$P(i,t) \qquad 1-(N-i)\,\lambda_a\,\Delta t \qquad 1-i\,\lambda_d\,\Delta t \tag{4.12}$$

$$P(i+1,t) \qquad 1-(N-i-1)\,\lambda_a\,\Delta t \qquad (i+1)\,\lambda_d\,\Delta t \tag{4.13}$$

The respective probabilities for the combinations (4.8), (4.9), and (4.10), are the respective products of the probabilities in (4.11), (4.12), and (4.13); and $P(i,t+\Delta t)$ is the sum of the terms thus obtained:

$$\begin{aligned} P(i,t+\Delta t) = {} & P(i-1,t)\,(N-i+1)\,\lambda_a\,\Delta t(1-(i-1)\,\lambda_d\,\Delta t) \\ & + P(i,t)\,(1-(N-i)\,\lambda_a\,\Delta t)\,(1-i\,\lambda_d\,\Delta t) \\ & + P(i+1,t)\,(1-(N-i-1)\,\lambda_a\,\Delta t)\,(i+1)\,\lambda_d\,\Delta t \\ & + o(\Delta t). \end{aligned} \tag{4.14}$$

$i = N.$

The following combinations will give N individuals under observation at the time $t + \Delta t$:

Number of individuals under observation at time t:	Arrivals	Departures	
	during interval Δt:		
$N - 1$	1	0	(4.15)
N	0	0	(4.16)

(and some other combinations for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$$P(N-1,t) \qquad \lambda_a\, \Delta t \qquad 1-(N-1)\,\lambda_d \Delta t \tag{4.17}$$

$$P(N,t) \qquad 1 \qquad 1-N\,\lambda_d \Delta t, \tag{4.18}$$

whence, by analogy with the above,

$$\begin{aligned} P(N,t+\Delta t) = {} & P(N-1,t)\,\lambda_a\,\Delta t(1-(N-1)\,\lambda_d\,\Delta t) \\ & + P(N,t)\,(1-N\,\lambda_d\,\Delta t) \\ & + o(\Delta t). \end{aligned} \tag{4.19}$$

From the equations (4.7), (4.14), and (4.19), we obtain the following differential equations for $\Delta t \to 0$:

$$P_t'(0,t) = -N\,\lambda_a\,P(0,t) + \lambda_d\,P(1,t) \tag{4.20}$$

$$\begin{aligned} P_t'(i,t) = {} & (N-i+1)\,\lambda_a\,P(i-1,t) - ((N-i)\,\lambda_a + i\,\lambda_d)\,P(i,t) \\ & + (i+1)\,\lambda_d\,P(i+1,t) \qquad 0 < i < N \end{aligned} \tag{4.21}$$

$$P_t'(N,t) = \lambda_a\,P(N-1,t) - N\,\lambda_d\,P(N,t). \tag{4.22}$$

As the initial value is $(j,\, t_0)$, the process — which we may call the binomial process — must also satisfy the corresponding initial condition.

If the process attains statistical equilibrium, we have

$$\sum_{i=0}^{N} P(i) = 1, \tag{4.23}$$

for which it is a necessary condition that the coefficients entering in the given differential equations are bounded.

The limiting distribution $P(i)$ is thus uniquely determined by the $(N+2)$ equations (4.20) — (4.22) and (4.23) in connexion with (3.21), since the rank of the system of equations thus obtained is $(N+1)$; hence

$$P(i) = \binom{N}{i}\left(\frac{\lambda_a}{\lambda_d+\lambda_a}\right)^{i}\left(\frac{\lambda_d}{\lambda_d+\lambda_a}\right)^{N-i}, \qquad i = 0, 1, \cdots, N, \tag{4.24}$$

or the binomial law with a probability depending, like the mean of the Poisson distribution (3.26), only on the relative values of λ_a and λ_d, and not on their absolute values. The absolute values play an important part, on the other hand, in the binomial process, the general solution of (4.20) — (4.22) satisfying the initial condition, as it will appear from the following.

It will sometimes be convenient to make use of the matrix symbolism[1]) in our discussions of the process of the binomial law.

[1]) See *e. g. Bohr & Mollerup* (1938—1942), *Aitken* (1944), *Bôcher* (1908).

A matrix A of order mn is a rectangular scheme of numbers or elements a_{ij} arranged in m rows and n columns:

$$A = \begin{Bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{Bmatrix}.$$

We use the brief symbol $A = \{a_{ij}\}$, and when we want to indicate the order of the matrix, we write A_{mn} instead of A. A_{nn} is a square matrix. E denotes a square matrix A_{nn} where $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$.

The sum of two matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ of the same order is the matrix $C = \{c_{ij}\} = \{a_{ij} + b_{ij}\}$.

If the matrix A has as many columns (n) as the matrix B has rows, the product matrix C of A and B is defined and given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj},$$

that is to say, $A_{mn}\, B_{nl} = C_{ml}$.

The transposed matrix $A^* = \{a^*_{ij}\}$ is a matrix of order nm such that its elements $a^*_{ij} = a_{ji}$.

To a square matrix A_{nn} corresponds a number A known as the determinant of the matrix A_{nn} such that

$$A = \sum_{(i,j,\dots,s)} a_{1i}\, a_{2j} \dots a_{ns}\, (-1)^{I(i,j,\dots,s)},$$

where $(i, j, \dots, s)$ is a permutation of the numbers $(1, 2, \dots, n)$ and $I(i, j, \dots, s)$ is the number of inversions.

A determinant retains its value if a linear combination of the other rows (columns) is added to one row (column).

The matrix $P(t,t_0)$ is $\{P(i,t \mid j,t_0)\}$. The matrix $P(t,t_0)$ is square, its order being $(N+1)(N+1)$ in the case of the binomial law discussed above.

$P'_t(t,t_0)$ denotes the matrix whose elements are differential quotients of the elements of $P(t,t_0)$, that is to say, $P'_t(t,t_0) = \{P'_t(i,t \mid j,t_0)\}$.

Using this symbolism, we may reduce the system of equations (4.20)—(4.22) to the form

$$P'_t(t,t_0) = AP(t,t_0), \tag{4.25}$$

where

$$A = \begin{Bmatrix} -N\lambda_a, & \lambda_d & , & 0 & , & 0 & ,\dots, & 0 & , & 0 & , & 0 \\ N\lambda_a, & -(N-1)\lambda_a - \lambda_d, & & 2\lambda_d & , & 0 & ,\dots, & 0 & , & 0 & , & 0 \\ 0 & , & (N-1)\lambda_a & , -(N-2)\lambda_d - 2\lambda_d, & & 3\lambda_d & ,\dots, & 0 & , & 0 & , & 0 \\ \vdots & & \vdots & & \vdots & & \vdots\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & , & 0 & , & 0 & , & 0 & ,\dots, & 2\lambda_a, & -\lambda_a - (N-1)\lambda_d, & & N\lambda_d \\ 0 & , & 0 & , & 0 & , & 0 & ,\dots, & 0 & , & \lambda_a & , -N\lambda_d \end{Bmatrix}, \tag{4.26}$$

or $A = \{a_{ij}\}$, where

$$
\begin{aligned}
a_{j-1,j} &= j\,\lambda_d, & j &= 1, \ldots, N,\\
a_{j,j} &= -(N-j)\,\lambda_a - j\,\lambda_d, & j &= 0, \ldots, N, \qquad (4.27)\\
a_{j+1,j} &= (N-j)\,\lambda_a, & j &= 0, \ldots, N-1,
\end{aligned}
$$

whereas all other elements are zero.

The initial condition may be written

$$P(t_0, t_0) = E. \qquad (4.28)$$

The solution of this system of differential equations having a matrix A which is independent of time can be written in the form

$$P(t, t_0) = \sum_{\nu=0}^{N} V_\nu\, U_\nu^*\, e^{r_\nu (t-t_0)} \qquad (4.29)$$

where V_ν and U_ν are matrices of order $(N+1)\ 1$ known as vectors, that is,

$$V_\nu = \begin{Bmatrix} V_{0\nu} \\ V_{1\nu} \\ \vdots \\ V_{N\nu} \end{Bmatrix} \qquad U_\nu = \begin{Bmatrix} U_{0\nu} \\ U_{1\nu} \\ \vdots \\ U_{N\nu} \end{Bmatrix} \qquad (4.30)$$

and where the *characteristic numbers* $r_0, r_1, r_2, \ldots, r_N$ are roots in the *secular equation* $R(x) = 0$ corresponding to the matrix

$$R(x) = A - x\,E, \qquad (4.31)$$

The form (4.29) is conditional in that we must have $r_\nu \neq r_\mu$ for $\nu \neq \mu$.

Evaluation of the characteristic numbers of the secular equation.

The characteristic numbers of the secular equation corresponding to (4.26) can be evaluated by simple row operations in $R(x)$; the operations will not change the value of the determinant.

The determinant $R_1(x)$ can be obtained from the determinant $R(x)$ by using,

as 1st row of $R_1(x)$, the 1st row of $R(x)$;
as 2nd row of $R_1(x)$, the sum of 1st and 2nd row of $R(x)$;
as 3rd row of $R_1(x)$, the sum of 1st, 2nd, and 3rd row of $R(x)$;
. . . .
. . . .
. . . .
as $(N+1)$th row of $R_1(x)$, the sum of 1st, 2nd, . . . , $(N+1)$th row of $R(x)$.

The determinant $R_2(x)$ is obtained from the determinant $R_1(x)$ by using

as 1st row of $R_2(x)$, *the 1st row of* $R_1(x)$;
as 2nd row of $R_2(x)$, *the sum of 1st and 2nd row of* $R_1(x)$;
as 3rd row of $R_2(x)$, *the sum of 1st, 2nd, and 3rd row of* $R_1(x)$;
. . . .
. . . .
. . . .
as Nth row of $R_2(x)$, *the sum of 1st, 2nd, ..., Nth row of* $R_1(x)$;
as (N+1)th row of $R_2(x)$, *the (N+1)th row of* $R_1(x)$.

The determinants $R_3(x)$, $R_4(x)$, ... are obtained similarly, the $(N+1)$th and Nth rows of $R_2(x)$ being retained in the next operation, and so on; accordingly the summation will stop automatically with $R_N(x)$.

In order to determine the elements of $R_N(x)$ we will employ a convenient auxiliary operation consisting in repeated summation of a sequence of $(N+1)$ elements where the mth element is unity and all others are zero.

No.	0	1	2 ...	m	$m+1$	$m+2$ ...	ν ...	$N-1$	N
Elements	0	0	0	1	0	0	0	0	0
1st sum	0	0	0	1	1	1	1	1	1
2nd sum	0	0	0	1	2	3	$\nu-m+1$	$N-m$	
3rd sum	0	0	0	1	3	6	$\binom{\nu-m+2}{2}$		
&c.									

Element no. ν in the μth sum will be

$$y_{\mu\nu}^{m} = \binom{\nu - m + \mu - 1}{\mu - 1}. \tag{4.32}$$

This auxiliary operation may be applied to the individual elements in the columns of $R(x)$. According to (4.27) in connexion with (4.31), the elements in the jth column are

$$\begin{aligned} a_{j-1,j} &= j\,\lambda_d, & j &= 1, \ldots, N, \\ a_{j,j} &= -(N-j)\,\lambda_a - j\,\lambda_d - x, & j &= 0, \ldots, N, \\ a_{j+1,j} &= (N-j)\,\lambda_a, & j &= 1, \ldots, N-1, \end{aligned} \tag{4.33}$$

and the others are zero.

By analogy with the foregoing, the elements in the jth column of $R_N(x) = |\,r_{ij}(x)\,|$ can be obtained by repeated summation of these $(N+1)$ elements, all of which are zero except the $(j-1)$th, the jth, and the $(j+1)$th. The 1st sum will be element no. N; the 2nd sum, element no.

$(N-1)$; the 3rd, no. $(N-2)$; ...; the $(N+1-i)$th, no. i. Similarly, we may take it that element no. i in the jth column $r_{ij}(x)$ is obtained as the sum of the corresponding sums of 3 columns where the $(j-1)$th, the jth, and the $(j+1)$th elements, respectively, are different from zero.

Then, using the result of the auxiliary operation (4.32), we have

$$r_{ij}(x) = j\,\lambda_d\, y^{j-1}_{N+1-i,\,i} + (-(N-j)\,\lambda_a - j\,\lambda_d - x)\, y^{j}_{N+1-i,\,i} + (N-j)\,\lambda_a\, y^{j+1}_{N+1-i,\,i} \tag{4.34}$$

Now, $R'(x) = \{r'_{ij}(x)\}$ can be obtained from $R_N(x)$ by the following row operation:

$$r'_{Nj}(x) = r_{Nj}(x)$$

$$r'_{ij}(x) = r_{ij}(x) - \frac{r_{i\,i+1}(x)}{r'_{i+1\,i+1}(x)}\, r'_{i+1\,j}(x) \qquad \begin{matrix} j = 0, \ldots, N \\ i = N-1, \ldots, 0. \end{matrix} \tag{4.35}$$

which will change neither the value of the determinant nor the values of the characteristic numbers.

When the reduction is completed, the elements of the determinant $R'(x)$ will be

$$r'_{ij}(x) = -\binom{N-j}{N-i}((\lambda_a + \lambda_d)(N-i) + x) \qquad \begin{matrix} j = 0, \ldots, N \\ i = 0, \ldots, N \end{matrix}, \tag{4.36}$$

since (4.36) is satisfied for $i = N$, as it will appear from (4.35), (4.34), and (4.32), and since insertion of (4.36) in the right-hand side of (4.35), using (4.34), gives

$$\begin{aligned} r'_{ij}(x) &= r_{ij}(x) - r_{i\,i+1}(x)\binom{N-j}{N-i-1}\frac{(\lambda_a+\lambda_d)(N-i-1)+x}{(\lambda_a+\lambda_d)(N-i-1)+x}, \\ &= j\,\lambda_d\binom{N-j+1}{N-i} - ((N-j)\,\lambda_a + j\lambda_d + x)\binom{N-j}{N-i} + (N-j)\,\lambda_a\binom{N-j-1}{N-i} \\ &\quad - \binom{N-j}{N-i-1}(i+1)\,\lambda_d \\ &= -\binom{N-j}{N-i}((\lambda_a+\lambda_d)(N-i)+x); \end{aligned}$$

this being (4.36), we have thus completed our inductive proof.

According to (4.36), all elements of the determinant $R'(x)$ are zero for $j > i$, and the value of the determinant is therefore the product of its diagonal elements:

$$R'(x) = \prod_{i=0}^{N} r'_{i,i}(x) = \prod_{i=0}^{N} (-((\lambda_a + \lambda_d)(N - i) + x)); \qquad (4.37)$$

hence it follows that the sought characteristic numbers are

$$r_\nu = -\nu(\lambda_a + \lambda_d) \qquad \nu = 0, 1, \ldots, N. \qquad (4.38)$$

In order to determine the vectors in (4.29), V_ν and U_ν, $\nu = 0, 1, \ldots, N$, we will now introduce (4.29) into the original equation (4.25), using the characteristic numbers (4.38). We get

$$\sum_{\nu=0}^{N} (A + \nu(\lambda_a + \lambda_d) E) V_\nu U_\nu^* e^{-\nu(\lambda_a + \lambda_d)(t - t_0)} = 0; \qquad (4.39)$$

since this must be satisfied for all values of t, it is necessary that

$$(A + \nu(\lambda_a + \lambda_d) E) V_\nu U_\nu^* = 0, \qquad \nu = 0, 1, \ldots, N. \qquad (4.40)$$

But this means that the vectors

$$(A + \nu(\lambda_a + \lambda_d) E) V_\nu = 0, \qquad \nu = 0, 1, \ldots, N. \qquad (4.41)$$

The matrix $A + \nu(\lambda_a + \lambda_d) E$ is of rank N, as all roots are single roots; hence it follows that each system of equations (4.41) with the unknown vectors V_ν has a solution depending on one parameter k_ν, and a single solution $\hat{V}_\nu$:

$$V_\nu = k_\nu \hat{V}_\nu. \qquad (4.42)$$

Now, if we introduce (4.29) into (4.28), the condition can be written in the form

$$\sum_{\nu=0}^{N} V_\nu U_\nu^* = E. \qquad (4.43)$$

Combined with (4.42) this gives

$$\sum_{\nu=0}^{N} \hat{V}_\nu k_\nu U_\nu^* = E. \qquad (4.44)$$

Putting $k_\nu U_\nu^* = \hat{U}_\nu^*$, (4.44) may be written

$$\hat{V} \hat{U}^* = E \qquad (4.45)$$

where $\hat{V}$ and $\hat{U}$ are matrices consisting of the original vectors $\hat{V}_\nu$ and $\hat{U}_\nu$ such that

$$\hat{V} = \{\hat{V}_0, \hat{V}_1, \ldots, \hat{V}_N\} \tag{4.46}$$

and

$$\hat{U} = \{\hat{U}_0, \hat{U}_1, \ldots, \hat{U}_N\}. \tag{4.47}$$

But $\hat{V}$ consists, by (4.41), of $(N + 1)$ mutually linearly independent vectors so that $\hat{V}$ has the rank $(N + 1)$. Thus there is one and only one solution of the system of equations (4.45).

Consequently, (4.25) has one and only one solution satisfying the condition (4.28). This solution is (4.29) with $V_\nu\ U_\nu^* = \hat{V}_\nu\ \hat{U}_\nu^*$, where $\hat{V}_\nu$ and $\hat{U}_\nu$ are determined by (4.41) and (4.45), and the characteristic numbers r_ν are determined by (4.31).

It follows from (4.29) and (4.38) that the process of the binomial law treated here has a limiting distribution for $t \to \infty$, *viz.*

$$\lim_{t \to \infty} P(t, t_0) = V_0\ U_0^*. \tag{4.48}$$

The matrix A being bounded, we have by (4.25) and (4.28) that

$$\sum_{i=0}^{N} P(i,j) = 1, \qquad j = 0, 1, \ldots, N, \tag{4.49}$$

or, using (4.48),

$$\sum_{i=0}^{N} v_{i,0}\ u_{j,0} = 1, \qquad j = 0, 1, \ldots, N, \tag{4.50}$$

from which it appears that $u_{j,0}$ is a constant not depending on j, and that $P(i,j)$ does not depend on j.

But this means that the binomial process when $t \to \infty$ attains statistical equilibrium, which is given by

$$A\ V_0 = 0 \tag{4.51}$$

combined with

$$\sum_{i=0}^{N} v_{i,0} = 1, \tag{4.52}$$

the result being (4.24).

In determining the binomial process we shall apply the method of solution stated above only to the case of $N = 1$, for which the matrix is defined by (4.26)

$$A = \begin{Bmatrix} -\lambda_a & \lambda_d \\ \lambda_a & -\lambda_d \end{Bmatrix} \tag{4.53}$$

According to (4.38), the characteristic numbers of the corresponding secular equation are

$$r_0 = 0, \quad r_1 = -(\lambda_a + \lambda_d). \tag{4.54}$$

V_0 and V_1 can be formed by means of the systems of equations (4.41) which may be written

$$\begin{Bmatrix} -\lambda_a & \lambda_d \\ \lambda_a & -\lambda_d \end{Bmatrix} \begin{Bmatrix} v_{0,0} \\ v_{1,0} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.55}$$

$$\begin{Bmatrix} \lambda_d & \lambda_d \\ \lambda_a & \lambda_a \end{Bmatrix} \begin{Bmatrix} v_{0,1} \\ v_{1,1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{4.56}$$

that is to say,

$$\begin{Bmatrix} v_{0,0} \\ v_{1,0} \end{Bmatrix} = \begin{Bmatrix} \lambda_d \\ \lambda_a \end{Bmatrix} k_0 \qquad \begin{Bmatrix} v_{0,1} \\ v_{1,1} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} k_1 . \tag{4.57}$$

Next, $\hat{U}$ can be found by means of (4.45) which may be written

$$\begin{Bmatrix} \lambda_d & 1 \\ \lambda_a & -1 \end{Bmatrix} \begin{Bmatrix} \hat{u}_{0,0} & \hat{u}_{0,1} \\ \hat{u}_{1,0} & \hat{u}_{1,1} \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}, \tag{4.58}$$

the solution of this system of equations being

$$\begin{Bmatrix} \hat{u}_{0,0} & \hat{u}_{0,1} \\ \hat{u}_{1,0} & \hat{u}_{1,1} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{\lambda_d + \lambda_a} & \dfrac{\lambda_a}{\lambda_d + \lambda_a} \\ \dfrac{1}{\lambda_d + \lambda_a} & \dfrac{-\lambda_d}{\lambda_d + \lambda_a} \end{Bmatrix} \tag{4.59}$$

The matrices $V_0 U_0^*$ and $V_1 U_1^*$ are thus

$$V_0 U_0^* = \begin{Bmatrix} \lambda_d \\ \lambda_a \end{Bmatrix} \begin{Bmatrix} \dfrac{1}{\lambda_d + \lambda_a}, & \dfrac{1}{\lambda_d + \lambda_a} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\lambda_d}{\lambda_d + \lambda_a} & \dfrac{\lambda_d}{\lambda_d + \lambda_a} \\ \dfrac{\lambda_a}{\lambda_d + \lambda_a} & \dfrac{\lambda_a}{\lambda_d + \lambda_a} \end{Bmatrix} \tag{4.60}$$

$$V_1 U_1^* = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \begin{Bmatrix} \dfrac{\lambda_a}{\lambda_d + \lambda_a}, & \dfrac{-\lambda_d}{\lambda_d + \lambda_a} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\lambda_a}{\lambda_d + \lambda_a} & \dfrac{-\lambda_d}{\lambda_d + \lambda_a} \\ \dfrac{-\lambda_a}{\lambda_d + \lambda_a} & \dfrac{\lambda_d}{\lambda_d + \lambda_a} \end{Bmatrix} \tag{4.61}$$

The complete solution of (4.25) and (4.28) is obtained by substitution of (4.54), (4.60), and (4.61), in (4.29):

$$P(t,t_0) = \begin{Bmatrix} \dfrac{\lambda_d}{\lambda_d + \lambda_a} & \dfrac{\lambda_d}{\lambda_d + \lambda_a} \\ \dfrac{\lambda_a}{\lambda_d + \lambda_a} & \dfrac{\lambda_a}{\lambda_d + \lambda_a} \end{Bmatrix} + \begin{Bmatrix} \dfrac{\lambda_a}{\lambda_d + \lambda_a} & \dfrac{-\lambda_d}{\lambda_d + \lambda_a} \\ \dfrac{-\lambda_a}{\lambda_d + \lambda_a} & \dfrac{\lambda_d}{\lambda_d + \lambda_a} \end{Bmatrix} e^{-(\lambda_a + \lambda_d)(t-t_0)} \tag{4.62}$$

or, in a more elaborate form,

$$\begin{aligned} P(0,t \mid 0,t_0) &= p_0 = \frac{\lambda_d}{\lambda_d + \lambda_a} + \frac{\lambda_a}{\lambda_d + \lambda_a}\, e^{-(\lambda_a + \lambda_d)(t-t_0)} \\ P(1,t \mid 0,t_0) &= q_0 = \frac{\lambda_a}{\lambda_d + \lambda_a} - \frac{\lambda_a}{\lambda_d + \lambda_a}\, e^{-(\lambda_a + \lambda_d)(t-t_0)} \\ P(0,t \mid 1,t_0) &= p_1 = \frac{\lambda_d}{\lambda_d + \lambda_a} - \frac{\lambda_d}{\lambda_d + \lambda_a}\, e^{-(\lambda_a + \lambda_d)(t-t_0)} \\ P(1,t \mid 1,t_0) &= q_1 = \frac{\lambda_a}{\lambda_d + \lambda_a} + \frac{\lambda_d}{\lambda_d + \lambda_a}\, e^{-(\lambda_a + \lambda_d)(t-t_0)}, \end{aligned} \tag{4.63}$$

from which the solution of (4.25) and (4.28) for any N can be derived. If j individuals out of the N possible are under observation at the time t_0, then there may be i individuals under observation at the time t, if ν out of the j are still under observation, and $i - \nu$ out of the $N - j$ individuals not under observation go over to the observation group; in the meantime, the $j - \nu$ individuals under observation have gone over to the group of individuals not under observation, whereas the $N - j - (i - \nu)$ individuals have remained unobserved. The probability for this, written with the notations used in (4.63), is

$$P(i,t \mid j,t_0) = \sum_{\nu=0}^{i} \binom{j}{\nu} q_1^{\nu}\, p_1^{j-\nu} \binom{N-j}{i-\nu} q_0^{i-\nu}\, p_0^{N-j-i+\nu} \tag{4.64}$$

which may also be written

$$P(i,t \mid j,t_0) = \frac{1}{i!} D^i_{z=0} (p_1 + q_1 z)^j\, (p_0 + q_0 z)^{N-j}. \tag{4.65}$$

It follows from (4.63) that

$$\begin{aligned} \lim_{t \to \infty} p_0 &= \lim_{t \to \infty} p_1 = p = \frac{\lambda_d}{\lambda_a + \lambda_d} \\ \lim_{t \to \infty} q_0 &= \lim_{t \to \infty} q_1 = q = \frac{\lambda_a}{\lambda_a + \lambda_d}, \end{aligned} \tag{4.66}$$

so that (4.64) leads to

$$\lim_{t \to \infty} P(i,t \mid j,t_0) = \binom{N}{i} q^i \, p^{N-i} \tag{4.67}$$

which is the binomial law (4.24).

It will appear from (4.63) and (4.64) that the absolute values of λ_a and λ_d are of consequence to the rapidity of the limit passage. For great intensities λ_a and λ_d, the limiting distribution will occur earlier than for small intensities; in other words, the limiting distribution will be established more quickly when there are many but short-lived arrivals than when there are few but long-lived arrivals.

The *mean number* of individuals being under observation at the time t is given by (4.64), putting $i = (i - \nu) + \nu$:

$$M_1 = \sum_{i=0}^{N} i \, P(i,t \mid j,t_0) = N\, q + (j - N\, q)\, e^{-(\lambda_a + \lambda_d)\,(t-t_0)}; \tag{4.68}$$

for $t \to \infty$, the mean value will change from j to the mean value $N\, q$ of the limiting distribution.

The *variance* at the time t in the distribution (4.64) is

$$V_1 = \sum_{i=0}^{N} (i - M_1)^2 \, P(i,t \mid j,t_0) = j p_1 q_1 + (N - j)\, p_0 q_0 \tag{4.69}$$

so that the dispersion passes from 0 to $\sqrt{N\, p\, q}$ as t passes from t_0 to ∞.

The mean value and dispersion in (4.68) and (4.69) are plotted for $N = 15$, $j = 12$, $\lambda_a = 0.0608$, $\lambda_d = 0.1216$, in fig. (4.1).

As the Poisson distribution may be interpreted as a binomial distribution where the number of individuals increases unrestrictedly, while the increase in the number of individuals under observation is fixed, the general solution of (3.17) and (3.18) satisfying the condition (3.19) will be obtainable as the limit of (4.64) for $N \to \infty$ and $(N - i)\, \lambda_a = \lambda'_a$.

Putting (4.64) in the form

$$P(i,t \mid j,t_0) = \sum_{\nu=0}^{i} \binom{j}{\nu} q_1^{\nu} p_1^{j-\nu} \frac{(N-j)^{i-\nu}}{(i-\nu)!} \left(\frac{\lambda_a}{\lambda_d + \lambda_a}\right)^{i-\nu} (1 - e^{-(\lambda_a + \lambda_d)\,(t-t_0)})^{i-\nu}$$
$$\left(\frac{\lambda_d}{\lambda_d + \lambda_a}\right)^{N-j-i+\nu} \left(1 + \frac{\lambda_a}{\lambda_d}\, e^{-(\lambda_a + \lambda_d)\,(t-t_0)}\right)^{N-j-i+\nu} \tag{4.70}$$

and letting $N \to \infty$, we obtain the limit

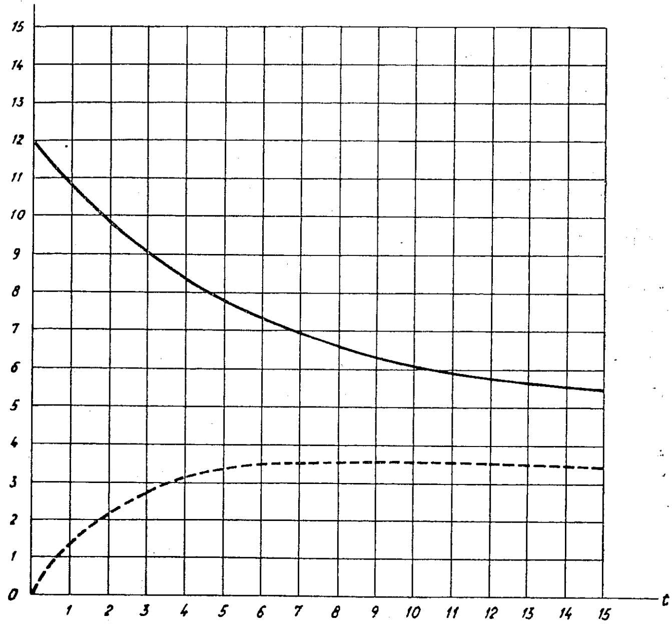


Fig. 4.1.

———— MEAN VALUE cf. (4.68)
- - - - - DISPERSION cf. (4.69)

$N = 15 \quad j = 12 \quad \lambda_a = 0{,}0608 \quad \lambda_d = 0{,}1216.$

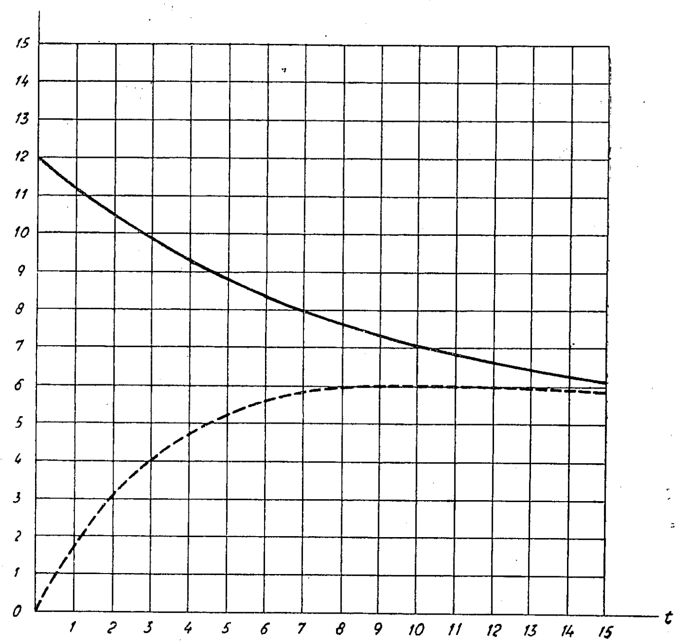


Fig. 4.2.

———— MEAN VALUE cf. (4.73)
- - - - - DISPERSION cf. (4.74)

$j = 12 \quad \lambda'_a = 0{,}6080 \quad \lambda_d = 0{,}1216.$

$$P(i,t \mid j,t_0) = \sum_{\nu=0}^{i} \binom{j}{\nu} e^{-\nu\lambda_d(t-t_0)} (1 - e^{-\lambda_d (t-t_0)})^{j-\nu} \tag{4.71}$$

$$\frac{1}{(i-\nu)!} \left(\frac{\lambda'_a}{\lambda_d}\right)^{i-\nu} (1 - e^{-\lambda_d(t-t_0)})^{i-\nu} e^{-\frac{\lambda'_a}{\lambda_d}} e^{-\frac{\lambda'_a}{\lambda_d} e^{-\lambda_d (t-t_0)}}$$

which may be written

$$P(i,t \mid j,t_0) = \tag{4.72}$$

$$\sum_{\nu=0}^{i} \frac{1}{(i-\nu)!} \left(\frac{\lambda'_a}{\lambda_d}\right)^{i-\nu} e^{-\frac{\lambda'_a}{\lambda_d}\left(1+e^{-\lambda_d (t-t_0)}\right)} \binom{j}{\nu} e^{-\nu\lambda_d (t-t_0)} (1 - e^{-\lambda_d (t-t_0)})^{j+i-2\nu},$$

which, for $\lambda'_a = \lambda_a$, is the general solution; for $t \to \infty$, the result is (3.26). The solution (4.72) has been derived by *Conny Palm*[1]) in a different manner.

The mean value and the variance in (4.72) can be obtained *e. g.* by making a limit passage in (4.68) and (4.69), the results being

$$M_2 = \frac{\lambda'_a}{\lambda_d} + \left(j - \frac{\lambda'_a}{\lambda_d}\right) e^{-\lambda_d (t-t_0)} \tag{4.73}$$

$$V_2 = \frac{\lambda_a{}'}{\lambda_d} + j e^{-\lambda_d (t-t_0)} (1 - e^{-\lambda_d (t-t_0)}). \tag{4.74}$$

These are plotted in fig. (4.2) for values corresponding to those of the previous example, with $\lambda'_a = N \lambda_a \dfrac{\lambda_d}{\lambda_a + \lambda_d}$.

5. *The Truncated Poisson Distribution. Erlang's Loss-Formula.*

The assumptions (3.1) and (3.2) stated in the chapter on Poisson's distribution do not suffice in quite a number of cases; in practice there will sometimes be an upper limit to the number n of individuals that can be observed simultaneously. This may be the case *e. g.* in problems concerning the conversation-carrying circuits in a junction group of n circuits, or the traffic-carrying lanes of a road with n lanes. The assumptions (3.1) and (3.2) must, in such cases, be supplemented with the following assumption:

At most n individuals can be observed simultaneously. Arrivals occurring during periods when n individuals are under observation shall be disregarded. (5.1)

1) See *C. Palm* (1943).

This is to be interpreted to the effect that admittance will be refused to new-comers (calls or vehicles) in the examples mentioned above.

The assumption (5.1) does not affect the differential equations (3.17) for $i = 0$ and (3.18) for $0 < i < n$.

$\underline{i = n.}$

The following combinations will give n individuals at the time $t + \Delta t$:

Number of individuals at time t:	Arrivals	Departures	
	during interval Δt:		
$n - 1$	1	0	(5.2)
n	0	0	(5.3)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$$P(n-1,t) \qquad \lambda_a \Delta t \qquad (1-(n-1)\lambda_d \Delta t) \tag{5.4}$$

$$P(n,t) \qquad 1 \qquad (1 - n\lambda_d \Delta t) \tag{5.5}$$

Hence it follows, in the usual manner, that

$$\begin{aligned} P(n,t+\Delta t) = & P(n-1,t)\,\lambda_a \Delta t(1-(n-1)\,\lambda_d \Delta t) \\ & + P(n,t)\,(1 - n\,\lambda_d \Delta t) \\ & + o(\Delta t), \end{aligned} \tag{5.6}$$

which leads to the differential equation

$$P'_t(n,t) = \lambda_a P(n-1,t) - n\,\lambda_d P(n,t) \tag{5.7}$$

The truncated Poisson process must thus satisfy the differential equations (3.17) for $i = 0$, (3.18) for $0 < i < n$, (5.7), and the initial condition.

If the process attains statistic equilibrium, we have

$$\sum_{i=0}^{n} P(i) = 1, \tag{5.8}$$

for which it is a sufficient condition that the coefficients entering in the given differential equations are bounded.

The limiting distribution will therefore be determinable from (3.17), (3.18) for $0 < i < n$, (5.7), (5.8), and (3.30). This system of equations has one and only one solution

$$P(i) = \frac{\dfrac{\left(\dfrac{\lambda_a}{\lambda_d}\right)^i}{i!}}{\sum\limits_{\nu=0}^{n} \dfrac{\left(\dfrac{\lambda_a}{\lambda_d}\right)^\nu}{\nu!}}. \tag{5.9}$$

The probability for observing the maximum number of individuals is obtained for $i = n$; all arrivals will be rejected throughout the duration of this state (all new calls will be lost):

$$P(n) = \frac{\dfrac{\left(\dfrac{\lambda_a}{\lambda_d}\right)^n}{n!}}{\sum\limits_{\nu=0}^{n} \dfrac{\left(\dfrac{\lambda_a}{\lambda_d}\right)^\nu}{\nu!}} \tag{5.10}$$

This expression is called Erlang's loss formula; applied to the above-mentioned example, a junction group of n circuits without any waiting device, it indicates the probability that a call cannot get through.

As in the previously mentioned cases, only the relative values of λ_a and λ_d are of consequence to the distribution function in statistical equilibrium.

The mean number of observed individuals (the mean number of occupied switches) is, by (5.9),

$$\sum_{i=0}^{n} i\, P(i) = \frac{\lambda_a}{\lambda_d} (1 - P(n)). \tag{5.11}$$

This, and not the amount $\frac{\lambda_a}{\lambda_d}$ of traffic offered, is the value that is found by measurements in practice.

6. The Truncated Binomial Distribution.

Just as there is a truncated Poisson process corresponding to the ordinary Poisson process, there is a truncated process corresponding to the binomial process treated in Chapter 4. There may be cases among the problems belonging under the assumptions of the binomial law, where at most n individuals can be observed simultaneously ($n < N$). This is the case *e. g.* where N subscribers, connected to a subscriber's cable containing n circuits, can carry on at most n conversations simultaneously. In such cases the assumptions (4.1) and (4.2) must be supplemented with the assumption (5.1).

This assumption does not affect the differential equations derived in (4.20) for $i = 0$ and in (4.21) for $0 < i < n$, but it affects the case of $i = n$.

$\underline{i = n.}$

The following combinations will give n individuals at the time $t + \Delta t$:

Number of individuals under observation at time t:	Arrivals	Departures	
	during interval Δt:		
$n - 1$	1	0	(6.1)
n	0	0	(6.2)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$$P(n-1,t) \qquad (N-n+1)\,\lambda_a \Delta t \qquad 1-(n-1)\,\lambda_d \Delta t \tag{6.3}$$

$$P(n,t) \qquad 1 \qquad 1-n\,\lambda_d \Delta t \tag{6.4}$$

Hence it follows in the usual manner that

$$\begin{aligned} P(n,t+\Delta t) = {} & P(n-1,t)\,(N-n+1)\,\lambda_a \Delta t(1-(n-1)\,\lambda_d \Delta t) \\ & + P(n,t)\,(1-n\,\lambda_d \Delta t) \\ & + o(\Delta t) \end{aligned} \tag{6.5}$$

which leads to the differential equation

$$P_t'(n,t) = (N-n+1)\,\lambda_a\,P(n-1,t) - n\,\lambda_d\,P(n,t) \tag{6.6}$$

which, combined with the differential equations (4.20) for $i = 0$ and (4.21) for $0 < i < n$ and the initial condition, defines the truncated binomial process.

If the process attains statistic equilibrium, we have

$$\sum_{i=0}^{n} P(i) = 1 \tag{6.7}$$

for which it is a sufficient condition that the coefficients entering in the given differential equations are bounded.

The limiting distribution, if any, will therefore be determinable from

(4.20), (4.21) for $0 < i < n$, (6.6), (6.7), and (3.30). This system of equations has one and only one solution

$$P(i) = \frac{\binom{N}{i}\left(\frac{\lambda_a}{\lambda_a+\lambda_d}\right)^i \left(\frac{\lambda_d}{\lambda_a+\lambda_d}\right)^{N-i}}{\sum_{\nu=0}^{n}\binom{N}{\nu}\left(\frac{\lambda_a}{\lambda_a+\lambda_d}\right)^\nu \left(\frac{\lambda_d}{\lambda_a+\lambda_d}\right)^{N-\nu}}, \quad i = 0, 1, \ldots, n, \tag{6.8}$$

which may be written in the form

$$P(i) = \frac{\binom{N}{i}\left(\frac{\lambda_a}{\lambda_d}\right)^i}{\sum_{\nu=0}^{n}\binom{N}{\nu}\left(\frac{\lambda_a}{\lambda_d}\right)^\nu}, \quad i = 0, 1, \ldots, n. \tag{6.9}$$

The probability for observing the maximum number of individuals is obtained for $i = n$, which — in the terms of the above mentioned example — corresponds to the probability for all circuits being occupied by conversations in a subscriber's cable containing n circuits and serving $N(> n)$ subscribers.

The mean number M of observed individuals (mean number of occupied circuits) is, by (6.8),

$$M = \sum_{i=0}^{n} i\, P(i) = N \frac{\lambda_a}{\lambda_a+\lambda_d} \frac{\sum_{\nu=0}^{n-1}\binom{N-1}{\nu}\left(\frac{\lambda_a}{\lambda_a+\lambda_d}\right)^\nu \left(\frac{\lambda_d}{\lambda_a+\lambda_d}\right)^{N-\nu}}{\sum_{\nu=0}^{n}\binom{N}{\nu}\left(\frac{\lambda_a}{\lambda_a+\lambda_d}\right)^\nu \left(\frac{\lambda_d}{\lambda_a+\lambda_d}\right)^{N-\nu}}; \tag{6.10}$$

this, and not the amount $N \frac{\lambda_a}{\lambda_a+\lambda_d}$ of traffic actually offered, is the value that is found by measurements in practice.

Some Applications of the Truncated Binomial Distribution.

Erlang applied the truncated binomial distribution especially to the elucidation of various problems in connexion with the above mentioned example where altogether $n(< N)$ switches are available to N subscribers, and all subscribers are supposed to have the same call traffic, consisting of incoming as well as outgoing calls.

The probability g_n that one, or more, of the n co-operating switches will be free is, by (6.9),

$$g_n = 1 - P(n) = \frac{\sum_{\nu=0}^{n-1} \binom{N}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}{\sum_{\nu=0}^{n} \binom{N}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu} \tag{6.11}$$

The probability g_N that a particular subscriber will be disengaged is

$$g_N = \sum_{i=0}^{n} \frac{N-i}{N} P(i) = \frac{\sum_{\nu=0}^{n} \binom{N-1}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}{\sum_{\nu=0}^{n} \binom{N}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}, \tag{6.12}$$

the probability that he is not included among the i subscribers engaged in conversation being $\frac{N-i}{N}$.

The probability g_{Nn} that a particular subscriber will be disengaged, and that there will be a free switch among the n switches in the group, is obtained by omitting the last term in (6.12), the result being

$$g_{Nn} = \sum_{i=0}^{n-1} \frac{N-i}{N} P(i) = \frac{\sum_{\nu=0}^{n-1} \binom{N-1}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}{\sum_{\nu=0}^{n} \binom{N}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}. \tag{6.13}$$

From (6.12) and (6.13) we may derive

$$g_N - g_{Nn} = \frac{N-n}{N} P(n) = \frac{\binom{N-1}{n} \left(\frac{\lambda_a}{\lambda_d}\right)^n}{\sum_{\nu=0}^{n} \binom{N}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}, \tag{6.14}$$

which is the probability that a given subscriber is disengaged and the group (all n switches) is occupied.

The probability that a disengaged subscriber cannot be called because all n switches in the group are occupied is

$$\frac{g_N - g_{N\,n}}{g_N} = 1 - \frac{g_{Nn}}{g_N} = \frac{\binom{N-1}{n} \left(\frac{\lambda_a}{\lambda_d}\right)^n}{\sum_{\nu=0}^{n} \binom{N-1}{\nu} \left(\frac{\lambda_a}{\lambda_d}\right)^\nu}, \tag{6.15}$$

which also represents the probability that the other $N-1$ subscribers are occupying the n switches in the group when the given subscriber wants to use his telephone.

7. Truncated Multidimensional Distributions.

In several cases Erlang employed simple truncated multidimensional distributions when investigating special loss problems. In addition to the cases treated by Erlang we shall include some other cases in the following; also, we shall make a detailed investigation of the scope of the method employed. The terminology used in discussing the problems is that of telephony, since all the chosen examples belong in this field; but this does not mean that other domains cannot turn such problems to account.

The assumptions (3.1) and (3.2) of the Poisson distribution are tacitly understood to be valid for the sources of traffic to be considered, unless directions to the contrary are expressly stated. Some of the problems involve the employment of assumptions corresponding to (5.1). For clearness' sake the following abbreviations are used:

$$\begin{aligned} b_{\nu i} &= \lambda_{a_i} \Delta t(1 - \nu\, \lambda_{d_i} \Delta t), & \beta_{\nu i} &= \lambda_{a_i}, \\ c_{\nu i} &= (1 - \lambda_{a_i} \Delta t)\,(1 - \nu\, \lambda_{d_i} \Delta t) & & \\ a_{\nu i} &= (1 - \lambda_{a_i} \Delta t)\, \nu\, \lambda_{d_i} \Delta t & \alpha_{\nu i} &= \nu\, \lambda_{d_i}, \end{aligned} \tag{7.1}$$

and the matrix corresponding hereto,

$$A\,(i) = \left\{ a_{r,\,s\,(i)} \right\} \tag{7.2}$$

where the elements

$$\begin{aligned} a_{s-1,\,s}(i) &= \alpha_{si}, & s &= 1, \ldots, n_i, \\ a_{s,\,s}(i) &= -\alpha_{si} - \beta_{si}, & s &= 0, \ldots, n_i, \\ a_{s+1,\,s}(i) &= \beta_{si}, & s &= 0, \ldots, n_i - 1, \end{aligned} \tag{7.3}$$

and the remaining elements are zero.

I.

The amount of traffic that requires handling between the exchanges A and H is y_1 erlang $\left(y_1 = \frac{\lambda_{a_1}}{\lambda_{d_1}}\right)$; the amount of traffic that requires handling between the exchanges B and H is y_2 erlang $\left(y_2 = \frac{\lambda_{a_2}}{\lambda_{d_2}}\right)$, cf. fig. 7.1. The probability for a desired change in the number of call connexions established between 2 exchanges does not depend on the number of call connexions established between the other 2 exchanges. The given traffic is offered to m switches (junction lines) on the stretch BA, and to n switches on the stretch AH. We seek the probability $P(\nu, \mu, t \mid r, s, t_0)$ that there will be ν connexions on the stretch AH and μ connexions on the stretch BAH at the time t, when there were r and s connexions, respectively, at the time t_0 previous to t. The switches in the 2 groups BA and AH are supposed to be co-operating as simple groups[1]).

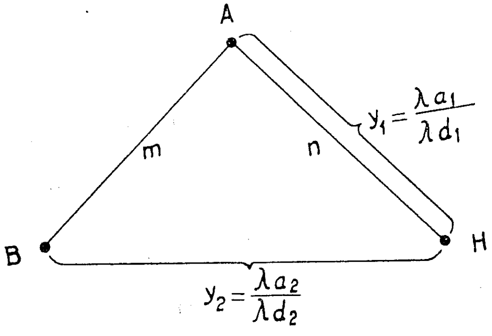


Fig. 7.1

The possible combinations of ν and μ are given by

$$\begin{aligned} 0 \leqq \quad & \nu \quad \leqq n \\ 0 \leqq \quad & \mu \quad \leqq m \\ 0 \leqq \nu & + \mu \leqq n \end{aligned} \tag{7.4}$$

as shown in fig. 7.2.

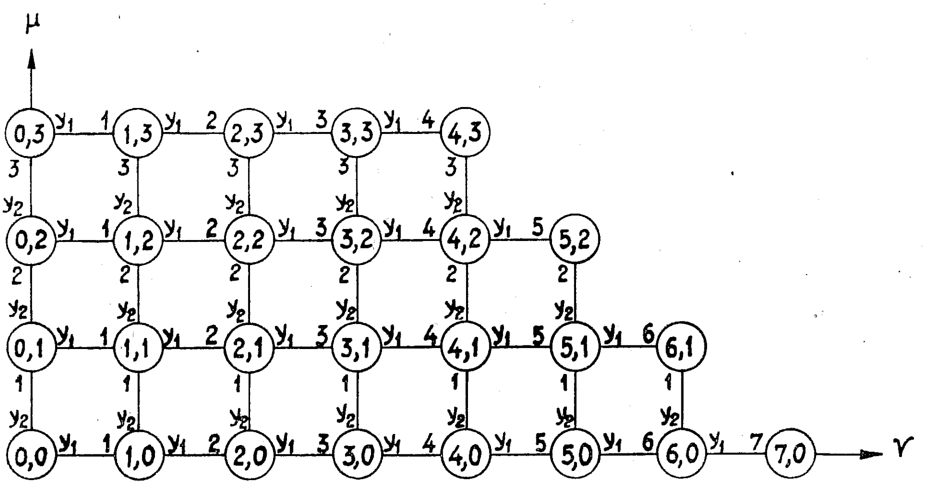


Fig. 7.2.

$P(\nu, \mu, t)$ can be determined by the same procedure as that followed hitherto. We find the probability $P(\nu, \mu, t + \Delta t)$, where $t_0 < t < t + \Delta t$, by means of the possible combinations of states at the time t and arrivals

[1]) The symbol $P(\nu, \mu, t)$ will be used instead of $P(\nu, \mu, t \mid r, s, t_0)$ where there is no risk of confusion.

and departures during the time interval Δt that will result in the state (ν, μ) at the time $t + \Delta t$. After a limit passage we derive a set of differential equations which, with some conditions, define the sought process.

Thus we obtain the following system of equations for the determination of $P(\nu, \mu, t + \Delta t)$:

$$\begin{aligned} P(\nu, \mu, t + \Delta t) = b_{\nu-1,1}\, b_{\mu-1,2}\, P(\nu - 1, \mu - 1, t) + b_{\nu-1,1}\, c_{\mu,2}\, P(\nu - 1, \mu, t) \\ + b_{\nu-1,1}\, a_{\mu+1,2}\, P(\nu - 1, \mu + 1, t) \\ + c_{\nu,1}\, b_{\mu-1,2}\, P(\nu, \mu - 1, t) + c_{\nu,1}\, c_{\mu,2}\, P(\nu, \mu, t) \\ + c_{\nu,1}\, a_{\mu+1,2}\, P(\nu, \mu + 1, t) \\ + a_{\nu+1,1}\, b_{\mu-1,2}\, P(\nu + 1, \mu - 1, t) + a_{\nu+1,1}\, c_{\mu,2}\, P(\nu + 1, \mu, t) \\ + a_{\nu+1,1}\, a_{\mu+1,2}\, P(\nu + 1, \mu + 1, t) \\ + o(\Delta t) \end{aligned} \tag{7.5}$$

which is valid for all points (ν, μ) inside the range (7.4), see fig. 7.2., and — with modifications corresponding to the results found for the truncated Poisson process and the truncated binomial process — for the marginal points, too.

Since

$$\lim_{\Delta t \to 0} \frac{P(\nu, \mu, t + \Delta t) - P(\nu, \mu, t)}{\Delta t} = P'_t(\nu, \mu, t), \tag{7.6}$$

(7.5) may, after the limit passage, be written (using the symbols defined in (7.1)):

$$\begin{aligned} P'_t(\nu, \mu, t) = \\ \beta_{\nu-1,1} P(\nu - 1, \mu, t) + \beta_{\mu-1,2} P(\nu, \mu - 1, t) - (\beta_{\nu,1} + \alpha_{\nu,1} + \beta_{\mu,2} + \alpha_{\mu,2}) P(\nu, \mu, t) \\ + \alpha_{\mu+1,2}\, P(\nu, \mu + 1, t) + \alpha_{\nu+1,1}\, P(\nu + 1, \mu, t) \end{aligned} \tag{7.7}$$

which may also be given the form

$$\begin{aligned} P'_t(\nu, \mu, t) = \\ \beta_{\nu-1,1}\, P(\nu - 1, \mu, t) - (\beta_{\nu,1} + \alpha_{\nu,1})\, P(\nu, \mu, t) + \alpha_{\nu+1,1}\, P(\nu + 1, \mu, t) \\ + \beta_{\mu-1,2}\, P(\nu, \mu - 1, t) - (\beta_{\mu,2} + \alpha_{\mu,2})\, P(\nu, \mu, t) + \alpha_{\mu+1,2}\, P(\nu, \mu + 1, t). \end{aligned} \tag{7.8}$$

The process must furthermore satisfy the conditional equation

$$P(\nu, \mu, t_0 \mid r, s, t_0) = \begin{cases} 1 & \text{for } (\nu, \mu) = (r, s) \\ 0 & \text{for } (\nu, \mu) \neq (r, s). \end{cases} \tag{7.9}$$

As

$$\sum_{\Omega_1}\sum P'_t(\nu,\mu,t) = 0, \tag{7.10}$$

where Ω_1 is the range stated in (7.4) and shown in fig. 7.2, it follows from (7.9), for n and m finite and $\beta_{\nu i}$ and $\alpha_{\nu i}$ bounded in accordance with (2.24), that

$$\sum_{\Omega_1}\sum P(\nu,\mu,t \mid r,s,t_0) \equiv 1. \tag{7.11}$$

If statistic equilibrium can occur, as expressed by

$$\lim_{t\to\infty} P(\nu,\mu,t \mid r,s,t_0) = P(\nu,\mu) \tag{7.12}$$

where not all $P(\nu, \mu) = 0$ and the limiting distribution does not depend on the initial value (r, s), it will follow from (7.7) and (7.12) that

$$\lim_{t\to\infty} P'_t(\nu,\mu,t \mid r,s,t_0) = 0 \tag{7.13}$$

and that (7.11) is also satisfied for the limiting distribution

$$\sum_{\Omega_1}\sum P(\nu,\mu) \equiv 1. \tag{7.14}$$

The limiting distribution is therefore defined by (7.7), (7.13), and (7.14).

The system of equations (7.7) with the condition (7.13) is satisfied by

$$P(\nu,\mu) = k_1 P_1(\nu)\, P_2(\mu) \tag{7.15}$$

where $P_1(\nu)$ and $P_2(\mu)$ are solutions of the systems of equations

$$0 = A_{(1)}\, P_1 \tag{7.16}$$

$$0 = A_{(2)}\, P_2 \tag{7.17}$$

where the matrices $A_{(1)}$ and $A_{(2)}$ are (7.2) for $n_1 = n$ and $n_2 = m$, and P_1 and P_2 are vectors whose respective elements are $P_1(0), \ldots, P_1(n)$ and $P_2(0), \ldots, P_2(m)$; the constant k_1 is determined from (7.14). But (7.16) and (7.17) are systems of equations determining one-dimensional limiting distributions of the truncated type. Under the assumptions chosen here, P_1 and P_2 are truncated Poisson distributions, so that the results of Chapter 5 respecting the truncated Poisson distribution are immediately applicable, and so we obtain

$$P(\nu,\mu) = k_1 \frac{y_1^\nu}{\nu!}\frac{y_2^\mu}{\mu!}. \tag{7.18}$$

Calls from exchange A to exchange H or *vice versa* will be lost if they are originated while

$$\nu + \mu = n \quad \text{for } \nu = n - m, n - m + 1, \ldots, n. \tag{7.19}$$

This state exists during a fraction of the time

$$E_{AH} = \sum_{\nu = n - m}^{n} P(\nu, n - \nu) \tag{7.20}$$

which thus indicates the probability that a call AH will be lost.

Calls from exchange B to exchange H or *vice versa* will be lost when

$$\mu = m, \qquad \nu = 0, 1, \ldots, n - m - 1,$$

and

$$\mu + \nu = n, \qquad \nu = n - m, n - m + 1, \ldots, n. \tag{7.21}$$

The probability for the occurrence of this state is

$$E_{BH} = \sum_{\nu=0}^{n-m-1} P(\nu, m) + \sum_{\nu = n-m}^{n} P(\nu, n - \nu). \tag{7.22}$$

The mean number of occupied switches in the group AH, which is equal to the handled amount of traffic, e_{AH}, measured in erlang, is

$$e_{AH} = \sum_{\Omega_1}\sum (\nu + \mu)\, P(\nu, \mu) = y_1(1 - E_{AH}) + y_2(1 - E_{BH}); \tag{7.23}$$

this is the value that is found by measurements on the group AH.

The mean number of occupied switches (the amount of traffic handled) in the group BA, e_{BA}, measured in erlang, is

$$e_{BA} = \sum_{\Omega_1}\sum \mu\, P(\nu, \mu) = y_2(1 - E_{BH}) \tag{7.24}$$

II.

We will now consider a case where the traffic from the exchanges A and B to the exchange H is directed over the transit exchange T, see fig. 7.3. The amount of traffic that requires handling between exchanges A and H is y_1 erlang; between exchanges B and H, y_2 erlang. There are n switches between exchanges A and T; m between B and T; p between T and H. The assumptions are the same as under I.

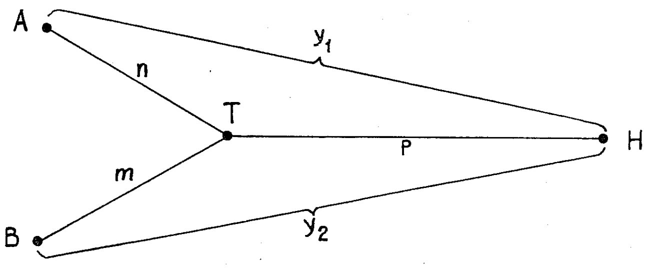


Fig. 7.3.

The possible combinations (ν, μ) of numbers of call connexions established between exchanges A and H and between exchanges B and H are bounded in the range Ω_2 as defined by

$$\begin{aligned} 0 \leqq \quad \nu \quad &\leqq n \\ 0 \leqq \quad \mu \quad &\leqq m \\ 0 \leqq \nu + \mu &\leqq p; \end{aligned} \tag{7.25}$$

hence quite naturally $n \leqq p$, $m \leqq p$, and $p \leqq n + m$, which we shall take for granted in the following even though it does not influence the results. The range Ω_2 is represented graphically in fig. 7.4.

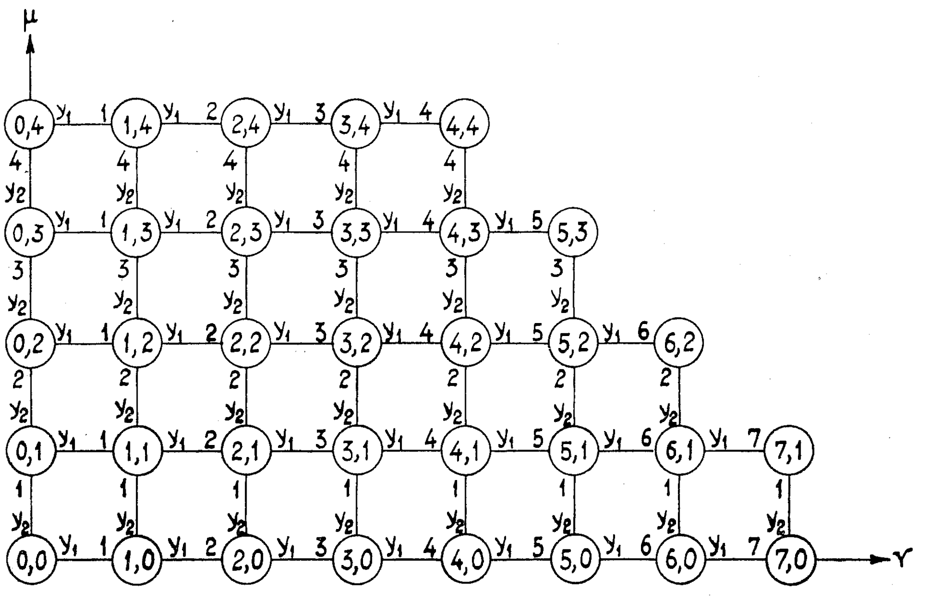


Fig. 7.4

Our present case differs from the preceding one only in the magnitude of the range Ω_2; the nature of Ω_1 and Ω_2 is such that neither will influence the method of demonstration of I essentially. The limiting distribution can therefore be derived from the results obtained in I, and so the determination of the constant is alone depending on the difference between the 2 ranges Ω_1 and Ω_2. Accordingly the limiting distribution will be

$$P(\nu, \mu) = k_2 \frac{y_1^\nu}{\nu!} \frac{y_2^\mu}{\mu!} \tag{7.26}$$

where the constant k_2 is determined from

$$\sum_{\Omega_2}\sum P(\nu, \mu) \equiv 1. \tag{7.27}$$

The stretch TH is fully occupied when

$$\nu + \mu = p \tag{7.28}$$

so that the probability that TH will be occupied is

$$E_{TH} = \sum_{\nu=p-m}^{n} P(\nu, p-\nu), \tag{7.29}$$

while calls from A to H will be lost when

$$\nu = n \quad \text{or} \quad \nu + \mu = p, \tag{7.30}$$

so that the probability that a call will be lost is

$$E_{AH} = \sum_{\mu=0}^{p-n-1} P(n, \mu) + \sum_{\mu=p-n}^{m} P(p-\mu, \mu). \tag{7.31}$$

Calls from B to H will be lost when

$$\mu = m \quad \text{or} \quad \nu + \mu = p \tag{7.32}$$

so that the probability that a call will be lost is

$$E_{BH} = \sum_{\nu=0}^{p-m-1} P(\nu, m) + \sum_{\nu=p-m}^{n} P(\nu, p-\nu). \tag{7.33}$$

In terms of erlang, the amounts of traffic handled over the stretches TH, TA, and TB are

$$e_{TH} = \sum_{\Omega_2}\sum (\nu + \mu) P(\nu, \mu) = y_1(1 - E_{AH}) + y_2(1 - E_{BH}) \tag{7.34}$$

$$e_{TA} = \sum_{\Omega_2}\sum \nu P(\nu, \mu) = y_1(1 - E_{AH}) \tag{7.35}$$

$$e_{TB} = \sum_{\Omega_2}\sum \mu P(\nu, \mu) = y_2(1 - E_{BH}). \tag{7.36}$$

These are the amounts of traffic that are found by measurements.

III.

The examples discussed under I and II may be extended to comprise a network of M exchanges connected to exchange H over the transit exchange T. The amount of traffic that requires handling between exchanges A_μ and exchanges H is $y_\mu = \dfrac{\lambda_{a_\mu}}{\lambda_{d_\mu}}$, the traffic being led over n_μ switches to the transit exchange T and from there over p switches to exchange H, or *vice versa*, see fig. 7.5.

It is natural to suppose that

$$n_\mu \leqq p, \qquad \mu = 1, \ldots, M, \tag{7.37}$$

and

$$p \leqq \sum_1^M n_\mu .$$

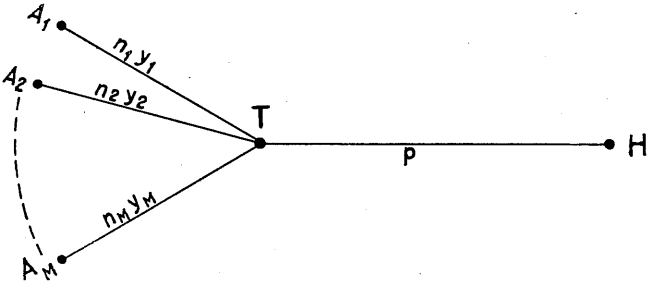


Fig. 7.5.

This extension leads to the following system of differential equations for the determination of $P(\nu_1, \nu_2, \ldots, \nu_M, t \mid r_1, r_2, \ldots, r_M, t)$

$$P'_t(\nu_1, \ldots, \nu_M, t) = \tag{7.38}$$

$$\beta_{\nu_1-1,1} P(\nu_1-1, \nu_2, \ldots, \nu_M, t) - (\beta_{\nu_1,1} + \alpha_{\nu_1,1}) P(\nu_1, \nu_2, \ldots, \nu_M, t) + \alpha_{\nu_1+1,1} P(\nu_1+1, \nu_2, \ldots, \nu_M, t)$$

$$+ \beta_{\nu_2-1,2} P(\nu_1, \nu_2-1, \nu_3, \ldots \nu_M, t) - (\beta_{\nu_2,2} + \alpha_{\nu_2,2}) P(\nu_1, \nu_2, \ldots, \nu_M, t) + \alpha_{\nu_2+1,2} P(\nu_1, \nu_2+1, \ldots, \nu_M, t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \beta_{\nu_M-1,M} P(\nu_1, \ldots, \nu_M-1, t) - (\beta_{\nu_M,M} + \alpha_{\nu_M,M}) P(\nu_1, \nu_2, \ldots, \nu_M, t) + \alpha_{\nu_M+1,M} P(\nu_1, \nu_2, \ldots, \nu_M+1, t)$$

which is valid for all points inside the range Ω_3 as defined by

$$0 \leqq \nu_i \leqq n_i \qquad i = 1, \ldots, M$$

$$0 \leqq \sum_1^M \nu_i \leqq p. \tag{7.39}$$

With simple modifications corresponding to the results obtained in the unidimensional cases, (7.38) is valid also for the marginal points.

If statistical equilibrium $P(\nu_1, \ldots, \nu_M)$ occurs, we have

$$P(\nu_1, \ldots, \nu_M) = k_3 \prod_{i=1}^{M} \frac{(y_i)^{\nu_i}}{(\nu_i)!} \tag{7.40}$$

where k_3 is determined, using the conditional equation corresponding to (7.9), by

$$\sum_{\Omega_3} \cdots \sum P(\nu_1, \ldots, \nu_M) = 1. \tag{7.41}$$

Calls from exchange A_i to exchange H will be lost when

$$\nu_i = n_i \quad \text{or} \quad \sum_{i=1}^{M} \nu_i = p, \tag{7.42}$$

the probability for this being

$$E_{A_iH} = \sum_{\Omega_3,\, \nu_i + \ldots + \nu_M = p} \cdots \sum P(\nu_1, \ldots, \nu_M) + \sum_{\Omega_3',\, \nu_i = n_i} \cdots \sum P(\nu_1, \ldots, \nu_M) \tag{7.43}$$

where Ω_3' denotes the range corresponding to Ω_3 that does not contain points for which

$$\sum_{j \neq i} \nu_j = p - n_i. \tag{7.44}$$

In terms of erlang, the amount of traffic handled over the stretch A_iT is

$$e_{A_iT} = y_i (1 - E_{A_iH}), \tag{7.45}$$

whereas

$$e_{HT} = \sum_{i=1}^{M} e_{A_iH} \tag{7.46}$$

is the traffic handled over the group of switches HT; it is this traffic that is estimated by measurements.

IV.

In the last example there was no local traffic between exchanges T and H; but there may be such local traffic, and, if so, we have a special case of III. An exchange A_i may then be regarded as identifiable with T, and thus $n_i = p$. Hence it follows that the range Ω_3' will be empty when exchange A_i is considered, so that only the first summation in (7.43) contributes to the probability of loss with respect to the traffic over TH $(= A_iH)$.

V.

The examples hitherto dealt with comprise only those cases where connexions can be established between a main exchange H and some other exchanges A_i over a transit exchange T, whereas local connexions between the exchanges A_i cannot be established over the same network. However, a perusal of the proofs will show that the demonstration itself does not preclude such mutual connexions over the same network if only the probability that a call will occur in one traffic channel during a given time Δt does not depend on the state of any of the other channels. Such an extension of the problem is of consequence only to the determination of the constant k which depends on the range Ω. An example of $(M+1)$ exchanges placed in the form of a star, as indicated in fig. 7.6, with at most $\binom{M+1}{2}$ interconnecting traffic channels of the type mentioned includes the examples discussed in the foregoing.

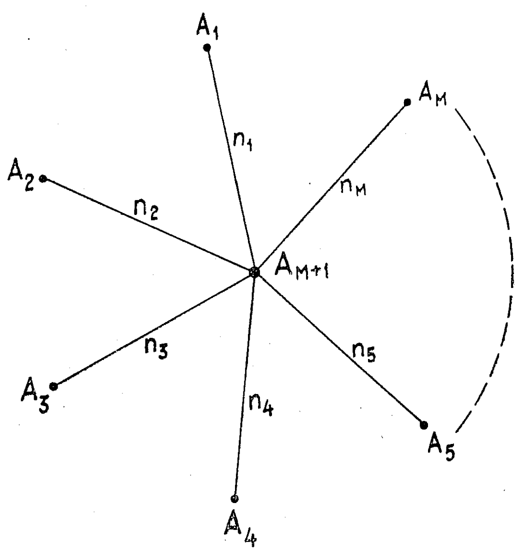


Fig. 7.6.

Between the exchanges A_{M+1} and the exchanges $A_1 \ldots A_M$ there are junction groups of $n_1 \ldots n_M$ switches which cooperate as simple groups. The total amount of bothway traffic offered to the channel connecting the exchanges A_i and A_j is $y_{ij} = y_{ji} = \dfrac{\lambda_{a_{ij}}}{\lambda_{d_{ij}}}$, where $y_{ii} = 0$.

The probability that there will be ν_{ij} call connexions established between the exchanges A_i and A_j when the system has attained statistical equilibrium is, for $j = 1, 2, \ldots, i-1$ and $i = 2, \ldots, M+1$,

$$P = P(\nu_{2,1}, \nu_{3,1}, \nu_{3,2}, \ldots, \nu_{M+1,1}, \ldots, \nu_{M+1,M}) = k \prod_{i=2}^{M+1} \prod_{j=1}^{i-1} \frac{(y_{ij})^{\nu_{ij}}}{(\nu_{ij})!} \tag{7.47}$$

where k is determined by

$$\sum \underset{\Omega}{\ldots} \sum P = 1. \tag{7.48}$$

By the assumption, the range Ω is defined by

$$0 \leqq \sum_{j=1}^{M+1} \nu_{ij} \leqq n_i, \qquad i = 1, 2, \ldots, M, \tag{7.49}$$

putting $\nu_{ij} = \nu_{ji}$ and $\nu_{ii} = 0$ in order to simplify the expressions.

Calls from exchange A_i to exchange A_j (where $i \neq j$), or *vice versa*, will be lost when

$$\sum_{j=1}^{M+1} \nu_{ij} = n_i, \qquad i \neq M + 1, \tag{7.50}$$

or when

$$\sum_{i=1}^{M+1} \nu_{ij} = n_j \qquad j \neq M + 1. \tag{7.51}$$

Calls from A_i to A_{M+1}, or *vice versa*, will be lost when

$$\sum_{j=1}^{M+1} \nu_{ij} = n_i \qquad i \neq M + 1 \tag{7.52}$$

which means that the probability that the stretch $A_i A_j$ will be fully occupied is

$$E_{A_iA_j} = \sum \underset{\Omega_1}{\ldots} \sum P + \sum \underset{\Omega_2 - \Omega_1}{\ldots} \sum P, \qquad i \neq j \neq M + 1, \tag{7.53}$$

where Ω_1 comprises combinations of ν_{ij} satisfying the conditions (7.49) and (7.50), while Ω_2 is defined by (7.49) and (7.51) so that $\Omega_2 - \Omega_1$ comprises those combinations inside Ω_2 which are not combinations inside Ω_1 as well.

The probability for lost calls between exchanges A_i and A_{M+1} is

$$E_{A_iA_{M+1}} = \sum \underset{\Omega_1}{\ldots} \sum P, \qquad i = 1, \ldots, M. \tag{7.54}$$

In terms of erlang, the amount of traffic handled over the channel $A_i A_j$ is

$$e_{ij} = y_{ij}\,(1 - E_{A_iA_j}), \tag{7.55}$$

whereas the channel $A_i A_{M+1}$ carries the traffic

$$e_i = \sum_{j=1}^{M+1} y_{ij}\,(1 - E_{A_iA_j}). \tag{7.56}$$

The total amount of traffic handled is

$$e = \sum_{i=2}^{M+1} \sum_{j=1}^{i-1} y_{ij} \left(1 - E_{A_i A_j}\right) \tag{7.57}$$

It will be noticed that the number of connexions to be established in A_{M+1} is, by (7.47),

$$n = \sum_{i=2}^{M+1} \sum_{j=1}^{i-1} \nu_{ij}. \tag{7.58}$$

VI.

The results obtained in V are applicable to a great number of special problems, such as the determination of the necessary number of connecting devices in a purely local exchange serving the internal traffic of a limited area only, *e. g.* a factory or a business house.

An investigation of such a private exchange plant consisting of a switchboard (without any connexions to the public exchanges) and some extension instruments, all of which are being used to the same extent, will, for a great number of extension instruments, furnish results approaching those expressed in Erlang's loss formula.

The results of V may also be used to determine the number of connecting devices required for the handling of local as well as incoming and outgoing calls in an ordinary switchboard.

They may further be used to determine, simultaneously, the necessary number of intercommunication possibilities, local and external in different directions, in an exchange, and the necessary number of switches in the groups connected to the exchange in so far as these are simple. It will generally be reasonable to apply *Moe's principle*[1]) to such calculations.

VII.

The ranges of occurrence Ω considered in the problems treated in this chapter have been defined by

$$0 \leqq \sum_{j=1}^{M+1} \nu_{ij} \leqq n_i, \qquad i = 1, \ldots, M.$$

It should be noticed that it is a necessary and sufficient condition for the obtained results that the range Ω is defined by

$$0 \leqq \sum_{j=1}^{M+1} m_j \, \nu_{ij} \leqq n_i \qquad i = 1, \ldots, M,$$

where n_i and m_j are positive integers or zero.

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[1]) See page 107.

The fact that only cases of traffic channels carrying Poisson-distributed traffic have been treated in this chapter should not be interpreted as signifying any restriction; many of the channels — or all of them — may be carrying binomially distributed traffic, but then the Poisson factors of the laws of distribution must be replaced with the corresponding binomial terms. This will change the meaning of the derived loss formula, however, and besides make the expressions of the amounts of traffic handled more complicated.

8. *Waiting Time Investigations. — Poisson's Distribution.*

The assumption (5.1) which is natural in the case of busy signal telephone systems must be modified to some extent in the case of waiting time systems. It was assumed in Chapter 5 that the number of simultaneously present individuals could not exceed n, and that arrivals occurring during periods when n individuals were already present, would be regarded as non-existent. But in systems where arrivals occurring during such periods are permitted to wait until a position or a connecting device becomes free, the assumption must be modified accordingly. We shall therefore in this chapter supplement the assumptions (3.1) and (3.2) with the following, which corresponds to (5.1):

The number of individuals that can be observed simultaneously is n at most. Arrivals occurring during periods when n individuals are already being observed are "put in a queue" and permitted to wait, in the order of their arrival, for "empty seats". (8.1)

The probability $P(i,t \mid j,t_0)$ to be found in the following expresses, for $i > n$, the probability that there will be n individuals under observation and $i - n$ waiting individuals at the time t.

The assumption (8.1) would not have affected our derivation of the differential equations (3.17) for $i = 0$ and (3.18) for $0 < i < n$; these must therefore be satisfied also by the process to be considered now, but it remains to be seen what happens for $i \geqq n$.

$\underline{i = n.}$

The following combinations will give n individuals at the time $t + \Delta t$:

Number of individuals at time t:	Arrivals (during interval Δt:)	Departures (during interval Δt:)	
$n - 1$	1	0	(8.2)
n	0	0	(8.3)
$n + 1$		1	(8.4)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(n-1,t)$	$\lambda_a \Delta t$	$1-(n-1)\,\lambda_d \Delta t$	(8.5)
$P(n,\,t)$	$(1-\lambda_a \Delta t)$	$1-n\,\lambda_d \Delta t$	(8.6)
$P(n+1,t)$	$(1-\lambda_a \Delta t)$	$n\,\lambda_d \Delta t.$	(8.7)

Hence it follows, as before, that

$$\begin{aligned} P(n,\,t+\Delta t) &= P(n-1,t)\,\lambda_a \Delta t(1-(n-1)\,\lambda_d \Delta t) \\ &+ P(n,t)\,(1-\lambda_a \Delta t)\,(1-n\,\lambda_d \Delta t) \\ &+ P(n+1,t)\,(1-\lambda_a \Delta t)\,n\,\lambda_d \Delta t \\ &+ o(\Delta t). \end{aligned} \tag{8.8}$$

<u>$i > n$.</u>

The following combinations will give i individuals at the time $t+\Delta t$:

Number of individuals at time t:	Arrivals (during interval Δt:)	Departures (during interval Δt:)	
$i-1$	1	0	(8.9)
i	0	0	(8.10)
$i+1$	0	1	(8.11)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(i-1,t)$	$\lambda_a \Delta t$	$1-n\,\lambda_d \Delta t$	(8.12)
$P(i,t)$	$1-\lambda_a \Delta t$	$1-n\,\lambda_d \Delta t$	(8.13)
$P(i+1,t)$	$1-\lambda_a \Delta t$	$n\,\lambda_d \Delta t.$	(8.14)

Hence it follows, as before, that

$$\begin{aligned} P(i,t+\Delta t) &= P(i-1,t)\,\lambda_a \Delta t(1-n\,\lambda_d \Delta t) \\ &+ P(i,t\,(1-\lambda_a \Delta t)\,(1-n\,\lambda_d \Delta t) \\ &+ P(i+1,t)\,(1-\lambda_a \Delta t)\,n\,\lambda_d \Delta t \\ &+ o(\Delta t). \end{aligned} \tag{8.15}$$

From (8.8) and (8.15) we obtain the differential equation

$$P'_t(i,t) = \lambda_a P(i-1,t) - (\lambda_a + n\,\lambda_d)\,P(i,t) + n\,\lambda_d P(i+1,t), \quad i \geqq n. \tag{8.16}$$

The process must satisfy (8.16) and also the initial condition.

If the process attains statistic equilibrium, we have

$$\sum_{i=0}^{\infty} P(i) = 1, \tag{8.17}$$

for which the condition (2.24) is satisfied, as

$$\alpha_\nu = \begin{cases} \nu\,\lambda_d, & \nu < n \\ n\lambda_d, & \nu \geqq n \end{cases} \quad \text{and} \quad \beta_\nu = \lambda_a. \tag{8.18}$$

Besides, it is a necessary condition that $\sum_{i=0}^{N} P(i,t)$ converges uniformly in N, which is satisfied if

$$\frac{\lambda_a}{n\,\lambda_d} \leqq k < 1. \tag{8.19}$$

The limiting distribution $P(i)$ is therefore determinable as the solution of the equations (3.17), (3.18), $0 < i < n$, (8.16) combined with (8.17) and (3.30), if (8.19) is satisfied.

The limiting distribution will be

$$P(i) = \begin{cases} k\left(\dfrac{\lambda_a}{\lambda_d}\right)^i \dfrac{1}{i!}, & i < n, \\[2ex] k\left(\dfrac{\lambda_a}{\lambda_d}\right)^i \dfrac{1}{n!\,n^{i-n}}, & i \geqq n, \end{cases} \tag{8.20}$$

where k is determined by (8.17).

The mean number of waiting individuals will be

$$\sum_{i=n}^{\infty} (i-n)\,P(i) = P(n)\,\frac{\dfrac{\lambda_a}{n\,\lambda_d}}{\left(1-\dfrac{\lambda_a}{n\,\lambda_d}\right)^2}. \tag{8.21}$$

The probability that all n "seats" will be occupied is

$$P(i \geqq n) = \sum_{i=n}^{\infty} P(i) = P(n)\,\frac{1}{1-\dfrac{\lambda_a}{n\,\lambda_d}}, \tag{8.22}$$

whereas the probability that there will be some individuals waiting is

$$\sum_{i=n+1}^{\infty} P(i) = P(n) \frac{\frac{\lambda_a}{n\,\lambda_d}}{1 - \frac{\lambda_a}{n\,\lambda_d}}. \tag{8.23}$$

The mean number of waiting individuals, if any, will be

$$\frac{\sum_{i=n+1}^{\infty} (i - n)\, P(i)}{\sum_{i=n+1}^{\infty} P(i)} = \frac{1}{1 - \frac{\lambda_a}{n\,\lambda_d}}, \tag{8.24}$$

and as the mean number of departures during the unit of time is $n\,\lambda_d$, we find the mean waiting time for the "queued-up" individuals to be

$$\frac{1}{1 - \frac{\lambda_a}{n\,\lambda_d}} \frac{1}{n\,\lambda_d} = \frac{1}{n\,\lambda_d - \lambda_a}. \tag{8.25}$$

The law of distribution for the waiting time, when the number of individuals waiting in the "queue" before the occurrence of the arrival under consideration is unknown, is

$$p(t)\, dt = (n\,\lambda_d - \lambda_a)\; e^{-(n\,\lambda_d - \lambda_a)\, t}\, dt; \tag{8.26}$$

if, on the other hand, the number of individuals in the "queue" is known, the distribution for the waiting time will be of the type given by (1.5).

When the arrivals not waiting are included, too, the mean waiting time will be

$$M_n = \frac{1}{n\,\lambda_d} \sum_{i=n}^{\infty} (i - n)\; P(i) = P(n) \frac{\lambda_a}{(n\,\lambda_d - \lambda_a)^2}, \tag{8.27}$$

where $\frac{1}{\lambda_d}$ is the mean holding time and λ_a is the mean number of arrivals during the unit of time.

9. *Waiting Time Investigations. — Binomial Distribution.*

The waiting time problem discussed in the preceding chapter was based upon the assumptions of the Poisson distribution, but there are other waiting time problems which can more advantageously be based upon the assumptions of the binomial distribution. When, in the field of telephony, a number of connecting devices is available to a limited group of subscribers only, the problem that faces us belongs to the latter class. In this and like cases the assumption (8.1) will be used in connexion with the assumptions (4.1) and (4.2) of the binomial distribution.

The sought probability $P(i,t \mid j,t_0)$ — which expression is to be interpreted in the same manner as in the preceding chapter — must here satisfy the differential equations (4.20) for $i = 0$ and (4.21) for $0 < i < n$, since these satisfy the conditions of the binomial law, and since the assumption (8.1) does not change the differential equations on this range. It must also satisfy the differential equations:

$$P_t'(i,t) = (N - i + 1)\lambda_a P(i-1,t) - ((N-i)\lambda_a + n\lambda_d)\,P(i,t) + n\lambda_d\,P(i+1,t), \qquad n \leqq i < N, \tag{9.1}$$

$$P_t'(N,t) = \lambda_a\,P(N-1,t) - n\,\lambda_d\,P(N,t), \tag{9.2}$$

which are derived in the same manner as (8.16), and it must furthermore satisfy the initial condition.

If the process attains statistic equilibrium, we have

$$\sum_{i=0}^{N} P(i) = 1, \tag{9.3}$$

for which it is a sufficient condition that the coefficients entering into the given differential equations are bounded.

The limiting distribution $P(i)$, which is determined by the equations (4.20), (4.21), (9.1), and (9.2), in connexion with (9.3) and (3.30), will therefore be

$$P(i) = \begin{cases} k\left(\dfrac{\lambda_a}{\lambda_d}\right)^i \dfrac{N^{(i)}}{i!}, & 0 \leqq i < n, \\[2ex] k\left(\dfrac{\lambda_a}{\lambda_d}\right)^i \dfrac{N^{(i)}}{n!\,n^{i-n}}, & n \leqq i \leqq N, \end{cases} \tag{9.4}$$

where the constant k is determined by (9.3), and $N^{(i)} = N(N-1)\ldots(N-i+1)$.

From this distribution may be derived expressions corresponding to those derived in the preceding chapter, (8.21) and (8.25).

The distribution for the waiting time is, as in Chapter 8, of the type indicated by (1.5) for a particular "item" in the "queue".

In this and the preceding chapter, the laws of distribution have been derived on the assumption that all arrivals occurring during periods of congestion will have to wait in the "queue" until they can get "seats". It is, however, very likely that some of those "standing in the queue" will give up waiting, thereby giving rise to extraordinary departures. This case was first suggested by *Conny Palm*[1]) who supposed that the probability for an individual's giving up waiting after having waited some time between t and $t + \Delta t$ is $\lambda_v \Delta t$; the distributions derived from this contain the waiting time distributions (8.20) and (9.4) found here, as well as the corresponding loss formulae (5.9) and (6.8).

10. Waiting Time Investigations. — Constant Holding Time.

The limiting distributions in the loss problems discussed in the foregoing depend on the mean holding time $\frac{1}{\lambda_d}$ only, and are otherwise independent of the distribution of the holding time[2]), whereas in waiting time problems the limiting distributions depend on the form of the law of distribution. Besides the waiting time problems of Chapters 8 and 9 where the distribution of the holding time is that indicated by (1.7), Erlang has also considered *a waiting time problem with constant holding time* (see (1.9)). His treatment of the problem — which was published in Matematisk Tidsskrift B, 1920 — shall be discussed in detail in the following; but first a couple of theorems ought to be mentioned which will be used as lemmata.

Lemma I. Jensen's theorem[3]) may be written

$$\sum_{j=0}^{\infty} e^{-(a+jx)} \frac{(a+jx)^j}{j!} = \frac{1}{1-x}, \quad \text{if} \quad \left| x e^{-x} \right| < \frac{1}{e}. \tag{10.1}$$

Introducing into this the notations $a = b\,\alpha_p$, $x = \alpha_p$, where

$$\alpha_p\, e^{-\alpha_p} = a\, e^{-a}\, e^{\frac{2\pi i p}{n}}, \qquad p = 0, 1, \ldots, n-1, \tag{10.2}$$

where i is the unit of imaginaries, we obtain

$$\sum_{j=0}^{\infty} (e^{-\alpha_p}\,\alpha_p)^{b+j} \frac{(b+j)^j}{j!} = \frac{\alpha_p^b}{1-\alpha_p}; \tag{10.3}$$

hence, by summing over p, we get

$$\sum_{p=0}^{n-1} \sum_{j=0}^{\infty} (e^{-\alpha_p}\,\alpha_p)^{b+j} \frac{(b+j)^j}{j!} = \sum_{p=0}^{n-1} \frac{\alpha_p^b}{1-\alpha_p} \tag{10.4}$$

which by insertion of (10.2) becomes

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1) *C. Palm* (1937).

2) See *e.g. C. Palm (loc. cit.)*.

3) *J. L. W. V. Jensen* (1902).

$$\sum_{j=0}^{\infty}{}' (e^{-a}\, a)^{b+j} \frac{(b+j)^j}{j!} \sum_{p=0}^{n-1}{}' e^{\frac{2\pi}{n} ip(b+j)}$$

$$= \sum_{j=0}^{\infty}{}' (e^{-a}\, a)^{b+j} \frac{(b+j)^j}{j!} \frac{1 - e^{2\pi i (b+j)}}{1 - e^{\frac{2\pi}{n} i (b+j)}} = \sum_{p=0}^{n-1}{}' \frac{a_p^b}{1 - a_p} \tag{10.5}$$

This sum thus contains only those terms for which $b + j = n\mu$, where $j \geqq 0$, that is to say,

$$\sum_{\mu \geqq \frac{b}{n}}^{\infty}{}' n\, (e^{-a}\, a)^{n\mu} \frac{(n\mu)^{n\mu - b}}{(n\mu - b)!} = \sum_{p=0}^{n-1}{}' \frac{a_p^b}{1 - a_p}, \tag{10.6}$$

so that

$$\sum_{\mu \geqq \frac{b}{n}}^{\infty}{}' e^{-a n \mu} \frac{(a n \mu)^{n\mu - b}}{(n\mu - b)!} = \frac{1}{n\, a^b} \sum_{p=0}^{n-1}{}' \frac{a_p^b}{1 - a_p} \tag{10.7}$$

where a_p is determined by (10.2). Erlang has arranged in tabular form, and represented graphically, the roots of the equation (10.2) $a_p = \zeta + i\,\eta$ for $n = 40$ and $a = 0,\ 0.1,\ 0.2,\ \ldots,\ 0.9$, and values for n which are divisors in 40, see Tables 14-15 on pp. 197-198 and Table 17 on p. 200.

Lemma II. It will appear from (1.4) that

$$\lim_{\mu \to \infty} P\{\, b\,\mu \geqq n\,\mu - \nu \,\} = 0 \tag{10.8}$$

where

$$P\{\, b\,\mu \geqq n\,\mu - \nu \,\} = \sum_{s = n\mu - \nu}^{\infty}{}' e^{-(\mu t - z)} \frac{(\mu\, t - z)^s}{s!} \tag{10.9}$$

when

$$z < \mu\, t, \quad t = a\, n < n, \quad 0 \leqq \nu < n. \tag{10.10}$$

The waiting time problem to be considered in this chapter is based upon the assumption (3.1) with respect to arrivals and upon the assumption (8.1) when all "seats" are occupied; the assumption with respect to departures will be this:

Any individual will leave after having been "seated" for the time t. (10.11)

This assumption is thus the only deviation from the assumptions of Chapter 8.

The sequence of points of time $t_1, t_2, t_3, \ldots, t_m$ at which arrivals occur is called a sequence of calls. In the following we shall measure time "backwards", using as time origin T_0 that particular point of time whose waiting time conditions we want to investigate.

Let the number of arrivals occurring during the time interval

$$(\mu - 1)\, t \leqq T < \mu\, t,\ \mu = 1, 2, \ldots, \tag{10.12}$$

be called a_μ, and let the number of arrivals occurring during the time interval

$$0 \leqq T < \mu\, t,\ \mu = 1, 2, \ldots. \tag{10.13}$$

be called b_μ.

If the time to be considered especially is $T_m = m\, t$ instead of T_0, the sequences of calls $a_{m\mu}$ and $b_{m\mu}$ corresponding to T_m can be derived from (10.12) and (10.13):

$$a_{m\mu} = a_{m+\mu}, \qquad \mu = 1, 2, \ldots, \tag{10.14}$$

$$b_{m\mu} = b_{m+\mu} - b_m, \quad \mu = 1, 2, \ldots, \tag{10.15}$$

where

$$b_0 = 0, \tag{10.16}$$

on which the number s of "seats" vacant at the time T_m out of the possible n "seats" (or connecting devices) depends.

If the individuals considered are telephone calls, we have that $n\, k$ calls, at most, can be started over n switches during a time $k\, t$, where k is an integer. If at least s switches shall be disengaged at the expiration of the time $k\, t$, then $(n\, k - s)$ is the maximum number of calls that may be started during the time $k\, t$. In order that there may be at least s free switches at the point of time T_m, this condition must be satisfied for all values of k, which means that

$$b_{m\mu} \leqq n\, \mu - s, \quad \mu = 1, 2, \ldots; \tag{10.17}$$

if this condition is satisfied, there will also be at least s free switches at the time T_m. Now, if as many of the calls as possible are "moved" as near to T_m as possible while (10.17) still has to be satisfied, we obtain

$$b_{m\mu} = n\, \mu - s, \quad \mu = 1, 2, \ldots. \tag{10.18}$$

Hence it follows that $a_{m1} = n - s$ and $a_{m\mu} = n$ for $\mu = 1, 2, \ldots$; but this means that there are exactly s free switches at T_m. It should be noticed that there will be exactly s free switches at the point of time T_m if only $b_{mk} = n\, k - s$ for one value of k, and (10.17) is otherwise satisfied, and there cannot be exactly s free switches if $b_{m\mu} < n\, \mu - s$ for all μ.

The concept of waiting time has not yet figured in these considerations. The waiting time, if any, for a call which has arrived at the time $T_m + z$ is not affected by the calls which may have arrived later ($T < T_m + z$), as the calls are put through in the order of their arrival. The condition that there will be at least s free switches at the time T_m available to a call arrived at the time $T_m + z$, is then still expressed by (10.17), except that new calls, if any, arriving during the time interval $T_m \leqq T < T_m + z$ must be excluded from the considered sequence of calls.

It follows from the assumption (3.1) that the number of arrivals r occurring during the time $k\,t$ follows Poisson's law of distribution with the mean $kt\lambda_a$ where λ_a is the intensity of arrivals. The probability for r arrivals is

$$p\,(r, k\,t\,\lambda_a) = \frac{(k\,t\,\lambda_a)^r}{r!}\,e^{-k\,t\,\lambda_a}, \tag{10.19}$$

while the probability that there will be more than r arrivals is

$$P\,(r, k\,t\,\lambda_a) = \sum_{\mu=r+1}^{\infty} p(\mu, k\,t\,\lambda_a). \tag{10.20}$$

The probability that (10.17) will be satisfied for $\mu = k$ is then

$$1 - P\,(k\,n - s, k\,t\,\lambda_a). \tag{10.21}$$

The limiting value of this is, in consequence of *Lemma II*, equal to 1 for $k \to \infty$, if $k\,n > k\,t\,\lambda_a$, which means that the amount of traffic measured in erlang must be less than the number n of switches. Hence it follows that, in making up which sequences of calls satisfy (10.17) and which do not, it is unnecessary to consider the sequences of calls for which it is true, for arbitrarily great values of M, that

$$b_{m\mu} > \mu\,n - s, \qquad \mu \geqq M, \tag{10.22}$$

since the probability for their occurrence converges to zero.

The other sequences of calls that do not satisfy (10.17) will have at least one element which satisfies one of the conditions:

$$b_{m\mu} = \mu\,n - \nu + n, \qquad \nu = s, s+1, \ldots, s+n-1. \tag{10.23}$$

As there, in consequence of (10.22), for each of these sequences of calls furthermore exists a number M for which

$$b_{m\mu} \leqq \mu\,n - s, \quad \text{for } \mu > M, \tag{10.24}$$

it follows that each of these sequences of calls has one and only one element b_{mk} which satisfies one of the conditions (10.23) for $\mu = k$ as well as the condition (10.24) for $M = k$. This means that the sequences of calls which do not satisfy (10.17), and which it is necessary to include in our list, may be grouped by means of the number k, so that

$$b_{mk} = kn - \nu + n, \qquad \nu = s, s+1, \ldots, s+n-1 \tag{10.25}$$

and $$b_{m\mu} \leqq \mu n - s, \qquad \mu > k.$$

Each of all the above mentioned sequences of calls will be included once and only once in the list when this method of grouping is employed.

Another way of writing (10.25) is

$$b_{mk} = kn - \nu + n, \qquad \nu = s, s+1, \ldots, s+n-1$$

and $$b_{m+kj} = b_{mk+j} - b_{mk} \leqq (k+j)\,n - s - kn + \nu - n \tag{10.26}$$

or $$b_{m+kj} \leqq jn + \nu - s - n, \qquad \nu = s, s+1, \ldots, s+n-1, \qquad j > 0.$$

The probability that there will be exactly s free switches at the time T_m among the possible n switches is called $p(s, n, T_m)$, and the probability that there will be at least s free switches at the time T_m is called

$$P(s, n, T_m) = \sum_{i=s}^{n} p(i, n, T_m). \tag{10.27}$$

The probability that a derived sequence of calls will not result in at least s free switches at the time T_m is, then, $1 - P(s, n, T_m)$. The probability corresponding to the sequences of calls included in the list is, according to (10.26), composed of terms indicating the probability that $(kn - \nu + n)$ arrivals will occur during the time kt (ranging from T_m to T_{m+k}), and, simultaneously, that there will be at least $n + s - \nu$ free switches at the time T_{m+k}, for $\nu = s, s+1, \ldots, s+n-1$ and $k = 1, 2, \ldots$.

This results in the following system of equations for the determination of $P(s, n, T_m)$:

$$1 - P(s, n, T_m) = \sum_{\nu=s}^{s+n-1} \sum_{k=1}^{\infty} p(kn - \nu + n, kt\lambda_a)\, P(n + s - \nu, n, T_{m+k}),$$
$$s = 1, \ldots, n, \qquad m = \ldots -1, 0, 1, 2, \ldots \tag{10.28}$$

where $p(kn - \nu + n, kt\lambda_a)$ is given by (10.19).

The probability that an arrival occurring at the time $T_m + z$ will have to wait at least the time z in order to find at least s free switches is called $1 - P(s, n, T_m, z)$, which expression, as previously mentioned, is also the probability that there will be at most $s - 1$ free switches at the time T_m, when arrivals occurring during the time interval $(T_m, T_m + z)$ are left out of account. The probability of a waiting time $< z$, as expressed by $P(s, n, T_m, z)$, is accordingly determinable from

$$1 - P(s, n, T_m, z) = \sum_{\nu=s}^{s+n-1} \sum_{k=1}^{\infty} p(kn - \nu + n, (kt - z)\,\lambda_a)\, P(n + s - \nu, n, T_{m+k}),$$
$$s = 1, \ldots, n, \qquad m = \ldots -1, 0, 1, 2, \ldots \tag{10.29}$$

If this process can attain statistic equilibrium, the limiting distribution $P(s, n, z)$ will be given by

$$1 - P(s, n, z) = \sum_{\nu=s}^{s+n-1} \sum_{k=1}^{\infty} p(kn - \nu + n, (kt - z)\lambda_a) P(n + s - \nu, n), \quad s = 1, \ldots, n, \tag{10.30}$$

where $P(s, n)$ is determined by (10.28) for $T_m \to \infty$, that is to say,

$$1 - P(s, n) = \sum_{\nu=s}^{s+n-1} P(n + s - \nu, n) \sum_{k=1}^{\infty} p(kn - \nu + n, kt\lambda_a), \quad s = 1, \ldots, n, \tag{10.31}$$

as compared with

$$P(0, n) = 1. \tag{10.32}$$

(10.30) may contain terms from a Poisson distribution with negative means, and Erlang has therefore, with a view to its practical applications, tabulated its values on a suitable range, as shown in Table 2, p. 137 and Table 13, pp. 195-196

It follows from *Lemma I* (10.7) that

$$\sum_{k=1}^{\infty} p(kn - \nu + n, kt\lambda_a) = \frac{1}{n\, a^{-n+\nu}} \sum_{p=0}^{n-1} \frac{a_p^{-n+\nu}}{1 - a_p} \quad \text{for } n < \nu \leqq 2n, \tag{10.33}$$

and that

$$\sum_{k=1}^{\infty} p(kn - \nu + n, kt\lambda_a) = \frac{1}{n\, a^{-n+\nu}} \sum_{p=0}^{n-1} \frac{a_p^{-n+\nu}}{1 - a_p} - p(n - \nu, 0) \quad \text{for } 0 < \nu \leqq n. \tag{10.34}$$

By inserting this in (10.31) we obtain

$$1 = \sum_{\nu=s}^{s+n-1} P(n + s - \nu, n) \frac{1}{n\, a^{-n+\nu}} \sum_{p=0}^{n-1} \frac{a_p^{-n+\nu}}{1 - a_p}, \quad s = 1, \ldots, n \tag{10.35}$$

which may be written in the form

$$1 = \sum_{p=0}^{n-1} \frac{1}{n(1 - a_p)} \left(\frac{a_p}{a}\right)^s \sum_{\nu=s}^{s+n-1} P(n + s - \nu, n) \left(\frac{a_p}{a}\right)^{-n+\nu-s}, \quad s = 1, \ldots, n \tag{10.36}$$

This system of equations is satisfied by the solution of the following equations:

$$\sum_{\nu=s}^{s+n-1} P(n + s - \nu, n) \left(\frac{a_p}{a}\right)^{-n+\nu-s} = \begin{cases} n(1 - a), & p = 0, \\ 0, & p = 1, \ldots, n-1. \end{cases} \tag{10.37}$$

Using the notations $\beta_p = \frac{a_p}{a}$ and $\beta_0 = 1$, these equations may be written

$$\sum_{\mu=1}^{n} P(\mu, n)\,(\beta_p)^{n-\mu} = \begin{cases} n(1-a) & p = 0, \\ 0, & p = 1, \ldots, n-1, \end{cases} \tag{10.38}$$

the solution of which is

$$P(\mu, n) = n(1-a)\,\frac{g(n-\mu+1)}{G(1)}, \quad \mu = 1, \ldots, n, \tag{10.39}$$

where

$$G(y) = y \prod_{\nu=1}^{n-1} (\beta_\nu - y) = \sum_{\nu=1}^{n} y^\nu g(\nu) \tag{10.40}$$

because

$$\sum_{\mu=1}^{n} n(1-a)\,\frac{1}{G(1)}\,g(n-\mu+1)\,\beta_p^{(n-\mu)} = n(1-a)\,\frac{1}{\beta_p}\,\frac{G(\beta_p)}{G(1)} = \begin{cases} n(1-a), & p=0, \\ 0, & p=1, \ldots, n-1. \end{cases} \tag{10.41}$$

From (10.39) is obtained

$$P(1, n) = n(1-a)\,\frac{1}{\prod_{\nu=1}^{n-1} (1-\beta_\nu)} \tag{10.42}$$

which expresses the probability for no waiting time.

A more elegant solution of this waiting time problem has been given by *C. D. Crommelin* in "The Post Office Electrical Engineers' Journal", April 1932, p. 41. Its leading idea shall be rendered here, since it seems to be much more in accordance with the reasoning of Erlang's other works than Erlang's own solution of this particular problem is.

The probability that there will be s occupied switches among the possible n circuits at the time T is called $p(s, n, T)$, while

$$P(s, n, T) = \sum_{r=0}^{s} p(r, n, T). \tag{10.43}$$

For $s > n$, the state s is interpreted as n occupied switches and $s - n$ waiting arrivals; otherwise the usual notations are employed.

The result: s occupied switches at the time $T = 0$, is obtained if at most n switches were occupied at the time $T = t$ and s arrivals have occurred during the interval $0 \leqq T < t$, or if n switches were occupied and one call was waiting at the time $T = t$ and $s - 1$ arrivals have occurred during the interval $0 \leqq T < t$, &c.

Hence it follows that

$$p(s, n, 0) = P(n, n, t)\, p(s, t\lambda_a) + p(n+1, n, t)\, p(s-1, t\lambda_a) + \cdots + p(n+s, n, t)\, p(0, t\lambda_a) \qquad \text{for } s = 0, 1, \ldots \tag{10.44}$$

If this system attains statistical equilibrium, the limiting distribution $p(s, n)$ can be determined from

$$p(s, n) = P(n, n)\, p(s, t\lambda_a) + \sum_{i=1}^{s} p(n+i, n)\, p(s-i, t\lambda_a), \quad s = 0, 1, \ldots. \tag{10.45}$$

If we use the generating function

$$f(y) \equiv \sum_{s=0}^{\infty} y^s p(s, n), \tag{10.46}$$

we can determine $f(y)$ by inserting (10.45) in (10.46); using (10.19) and reducing the expressions we obtain

$$f(y) = \frac{Q_n(y) - y^n P(n, n)}{1 - y^n e^{t\lambda_a(1-y)}}, \tag{10.47}$$

where $Q_n(y) = \sum_{s=0}^{n} y^s p(s, n)$.

An investigation of the roots of the denominator in (10.47) will show that the function may be written

$$f(y) = \frac{-n(1-a)}{\prod_{i=1}^{n-1}(1-\beta_i)} \cdot \frac{\prod_{i=0}^{n-1}(y-\beta_i)}{1 - y^n e^{t\lambda_a(1-y)}}. \tag{10.48}$$

The generating function for $P(s, n)$,

$$F(y) \equiv \sum_{s=0}^{\infty} y^s P(s, n), \tag{10.49}$$

can be derived from $f(y)$, as

$$F(y)(1-y) = f(y); \tag{10.50}$$

from this and (10.48) it follows that

$$P(n-1, n) = \frac{n(1-a)}{\prod_{i=1}^{n-1}(1-\beta_i)}, \tag{10.51}$$

which denotes the probability for no waiting time.

In order to determine the probability of a waiting time less than $z = mt + \tau$, where m is an integer and $\tau < t$, *Crommelin* introduces the probability b_r that at most r among a number of calls that have been established or are waiting at some point of time or other, will not have come to an end τ units of time later. Similarly, b_{nm+n-1} denotes the probability that at most $nm + n - 1$ among a number of new calls (arrivals) that have arrived before the point of time under consideration, will not have been terminated τ units of time later, *i. e.*, there will be at most $n - 1$ calls left $mt + \tau$ units of time later. This means that b_{nm+n-1} denotes the probability for a waiting time less than $m\,t + \tau$:

$$b_{nm+n-1} = P(< mt + \tau). \tag{10.52}$$

The generating function for b_r,

$$G(y) \equiv \sum_{\nu=0}^{\infty} b_\nu y^\nu, \tag{10.53}$$

is determined by observing that

$$P(r, n) = \sum_{\nu=0}^{r} b_\nu p(r - \nu, \tau\lambda_a) \tag{10.54}$$

which, in connexion with (10.49) and (10.53), leads to the result:

$$F(y) = e^{\tau\lambda_a(y-1)}\, G(y). \tag{10.55}$$

It should be noticed that

$$F(y) = G(y) \tag{10.56}$$

for $\tau = 0$, which means that

$$P(< mt) = b_{nm+n-1} = P(nm + n - 1, n). \tag{10.57}$$

Using these results *Crommelin* arrives at the following expression for the mean waiting time M:

$$M = \frac{1}{na} \sum_{p=1}^{n-1} \frac{1}{1 - \beta_p} + \frac{na^2 - (n - 1)}{2\,n\,a(1 - a)}, \tag{10.58}$$

the unit of time being equal to the holding time.

Erlang published, in the same form, his results concerning the mean waiting time in the special cases of $n = 1$, 2, and 3; he had, however, already at that time made some notes containing the general formula (10.58) which he had worked out in collaboration with *H. Cl. Nybølle.*

11. Combined Waiting Time and Loss Problems.

Whereas the truncated distributions and waiting time distributions mentioned in the preceding chapters are applicable only to those problems where "new" calls, finding all switches occupied, are either lost or "put in a queue", there are other problems where such calls normally are "put in a queue", but may be lost under certain conditions. The older types of manual telephone exchanges provide an example of the latter kind; in such exchanges, groups of n operators would be cooperating when handling the calls so that calls coming to the group while all n operators were busy were allowed to wait for an operator to be disengaged if not m previously arrived calls were waiting already; if this was the case, the new calls would be referred to some other group with a disengaged operator. In manual exchanges of newer type with automatic distribution of calls each group consists of 1 operator, and new calls will be "lost" if m previously arrived calls are waiting already, regardless of whether there are disengaged operators or not. There are other systems, again, where new calls are permitted to wait in an operator's position only when, say, $N-1$ of the total number N of operators are busy working calls, but will be lost if m previously arrived calls are waiting in that position already.

Erlang took the first two of these examples up for treatment; he worked out his solution of the problems on the assumption that there is a large number of groups and that the holding time follows the distribution stated in (1.7).

In the case of the waiting time problem contained in the first mentioned example where surplus calls are referred to a disengaged operator, an arrival may occur in two ways: it may be a new call, or it may be a call transferred from some other group. *The new calls are supposed to comply with the assumption* (3.1); as to the transferred calls, we have:

The probability for the occurrence of a transferred call is asymptotically proportional to the length of the time interval under consideration and to the number of free positions (*operators*) *in the group.* (11.1)

The factor of proportionality is called $\lambda_{a'}$, and *the departures* caused by the termination of calls *follow the assumption* (3.2) without any modifications.

These assumptions must be supplemented with the assumption (8.1) *for the waiting time and the assumption* (5.1) *respecting loss, n* being replaced by $n+m$ in the latter.

In order to determine $P(i,t \mid j,t_0)$, which denotes the probability that there will be altogether i individuals under observation — or waiting — in a group at the time t when there were j individuals at the time t_0 previous to t, we consider as usual the time interval $(t_0, t + \Delta t)$ and make a limit passage.

The derivation of the necessary differential equations is, as before, to a certain extent dependent on the magnitude of the number of individuals.

$i = 0$.

The following combinations will result in 0 individuals at the time $t + \Delta t$:

Number of individuals at time t:	New calls	Transferred calls	Terminated calls	
		during time interval Δt:		
0	0	0	0	(11.2)
1	0	0	1	(11.3)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(0, t)$	$(1 - \lambda_a \Delta t)$	$(1 - n\, \lambda_{a'} \Delta t)$	1	(11.4)
$P(1, t)$	$(1 - \lambda_a \Delta t)$	$(1 - (n - 1)\, \lambda_{a'} \Delta t)$	$\lambda_d \Delta t$,	(11.5)

whence it follows that

$$P(0, t + \Delta t) = P(0,t)\,(1 - \lambda_a \Delta t)\,(1 - n\,\lambda_{a'} \Delta t) + P(1,t)\,(1 - \lambda_a \Delta t)\,(1 - (n - 1)\,\lambda_{a'} \Delta t)\,\lambda_d \Delta t + o(\Delta t). \tag{11.6}$$

$0 < i < n$.

The following combinations will result in i individuals at the time $t + \Delta t$:

Number of individuals at time t:	New calls	Transferred calls	Terminated calls	
		during time interval Δt:		
$i - 1$	1	0	0	(11.7)
$i - 1$	0	1	0	(11.8)
i	0	0	0	(11.9)
$i + 1$	0	0	1	(11.10)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(i-1,t)$	$\lambda_a \Delta t$	$1-(n-i+1)\,\lambda_{a'}\,\Delta t$	$1-(i-1)\,\lambda_d \Delta t$	(11.11)
$P(i-1,t)$	$1-\lambda_a \Delta t$	$(n-i+1)\,\lambda_{a'}\,\Delta t$	$1-(i-1)\,\lambda_d \Delta t$	(11.12)
$P(i,t)$	$1-\lambda_a \Delta t$	$1-(n-i)\,\lambda_{a'}\,\Delta t$	$1-i\,\lambda_d \Delta t$	(11.13)
$P(i+1,t)$	$1-\lambda_a \Delta t$	$1-(n-i-1)\,\lambda_{a'}\,\Delta t$	$(i+1)\,\lambda_d \Delta t$	(11.14)

whence it follows that

$$\begin{aligned} P(i,t+\Delta t) = {} & P(i-1,t)\lambda_a\Delta t(1-(n-i+1)\lambda_{a'}\Delta t)\,(1-(i-1)\lambda_d\Delta t) \\ & +P(i-1,t)(1-\lambda_a\Delta t)(n-i+1)\lambda_{a'}\Delta t(1-(i-1)\lambda_d\Delta t) \\ & +P(i,t)(1-\lambda_a\Delta t)(1-(n-i)\lambda_{a'}\Delta t)(1-i\lambda_d\Delta t) \\ & +P(i+1,t)(1-\lambda_a\Delta t)(1-(n-i-1)\lambda_{a'}\Delta t)(i+1)\lambda_d\Delta t \\ & +o(\Delta t). \end{aligned} \tag{11.15}$$

$\underline{i = n.}$

The following combinations will result in n individuals at the time $t+\Delta t$:

Number of individuals at time t:	New calls	Transferred calls	Terminated calls	
	during time interval Δt:			
$n-1$	1	0	0	(11.16)
$n-1$	0	1	0	(11.17)
n	0	0	0	(11.18)
$n+1$	0	0	1	(11.19)

(and some other combinations, for which the probabilities are asymptotically equal to zero).

The probabilities corresponding to the above cases are

$P(n-1,t)$	$\lambda_a \Delta t$	$1-\lambda_{a'}\Delta t$	$1-(n-1)\,\lambda_d \Delta t$	(11.20)
$P(n-1,t)$	$1-\lambda_a \Delta t$	$\lambda_{a'}\Delta t$	$1-(n-1)\,\lambda_d \Delta t$	(11.21)
$P(n,t)$	$1-\lambda_a \Delta t$	1	$1-n\,\lambda_d \Delta t$	(11.22)
$P(n+1,t)$	$1-\lambda_a \Delta t$	1	$n\,\lambda_d \Delta t$	(11.23)

whence it follows that

$$\begin{aligned} P(n,t+\Delta t) = {} & P(n-1,t)\,\lambda_a\Delta t(1-\lambda_{a'}\,\Delta t)\,(1-(n-1)\,\lambda_d\Delta t) \\ & + P(n-1,t)\,(1-\lambda_a\Delta t)\,\lambda_{a'}\,\Delta t\,(1-(n-1)\,\lambda_d\Delta t) \\ & + P(n,t)\,(1-\lambda_a\Delta t)\,(1-n\,\lambda_d\Delta t) \\ & + P(n+1,t)\,(1-\lambda_a\Delta t)\,n\,\lambda_d\Delta t \\ & + o(\Delta t). \end{aligned} \tag{11.24}$$

$n < i < n + m.$

The assumptions employed in this case agree with the assumptions in Chapter 8 leading to the differential equations (8.16) for $n < i < m$.

$i = n + m.$

This case corresponds to that treated in (5.2)—(5.6), except that the number of individuals n is here replaced by $n + m$ and the probabilities for terminated calls are $(1 - n\,\lambda_d \Delta t)$. We obtain the following expression corresponding to (5.6):

$$P(n + m, t + \Delta t) = P(n + m - 1, t)\,\lambda_a \Delta t (1 - n\,\lambda_d \Delta t) + P(n + m, t)\,(1 - n\,\lambda_d \Delta t) + o(\Delta t). \tag{11.25}$$

$P(i,t)$ must thus satisfy the following system of differential equations derived from (11.6), (11.15), (11.24), (11.25), and (8.16):

$$P_t'(0,t) = -(\lambda_a + n\,\lambda_{a'})\,P(0,t) + \lambda_d P(1,t) \tag{11.26}$$

$$P_t'(i,t) = (\lambda_a + (n - i + 1)\,\lambda_{a'})\,P(i-1,t) - (\lambda_a + (n - i)\,\lambda_{a'} + i\,\lambda_d)\,P(i,t) + (i+1)\,\lambda_d P(i+1,t), \qquad 0 < i < n, \tag{11.27}$$

$$P_t'(n,t) = (\lambda_a + \lambda_{a'})\,P(n-1,t) - (\lambda_a + n\lambda_d)\,P(n,t) + n\lambda_d P(n+1,t) \tag{11.28}$$

$$P_t'(i,t) = \lambda_a P(i-1,t) - (\lambda_a + n\,\lambda_d)\,P(i,t) + n\,\lambda_d P(i+1,t), \qquad n < i < n + m, \tag{11.29}$$

$$P_t'(n+m,t) = \lambda_a P(n + m - 1, t) - n\,\lambda_d P(n+m,t), \tag{11.30}$$

and the initial condition.

If the considered process attains statistic equilibrium we have

$$\sum_{i=0}^{n+m} P(i) = 1, \tag{11.31}$$

so that the limiting distribution $P(i)$ can be determined as the solution of the equations (11.26)—(11.31) in connexion with (3.30). This system of equations has one and only one solution, *viz.*,

$$P(i) = \begin{cases} k \dfrac{\prod\limits_{\nu=0}^{i-1} \left(\dfrac{\lambda_a}{\lambda_d} + (n-\nu)\dfrac{\lambda_{a'}}{\lambda_d}\right)}{i!}, & i \leqq n, \\ P(n) \left(\dfrac{\lambda_a}{n\,\lambda_d}\right)^{i-n}, & n < i \leqq n+m, \end{cases} \tag{11.32}$$

where k is determined by (11.31).

$P(n+m)$ is of special importance as it indicates the probability that a call arriving in the group will be lost or transferred.

If all the groups of an exchange enter into statistic equilibrium, the number of lost "new" calls will be equal to the number of incoming transferred calls, or

$$P(n+m)\,\lambda_a = \lambda_{a'}\left(n - \frac{\lambda_a}{\lambda_d}\right), \tag{11.33}$$

which in connexion with (11.32) serves to determine $\lambda_{a'}$.

The waiting time distribution can be derived from (11.32), using (1.3).

The probability that a call arriving while the state i is prevailing will have to wait more than the time z is thus

$$P(>z \mid i) = e^{-zn\lambda_d} \sum_{\nu=0}^{i-n} \frac{(zn\,\lambda_d)^\nu}{\nu!}, \quad i \geqq n, \tag{11.34}$$

whereas the probability for a waiting time $> z$ for any call is

$$P(>z) = \sum_{i=n}^{n+m-1} P(i)\,P(>z \mid i), \tag{11.35}$$

In the two cases the mean waiting time will be

$$M(i) = \int_0^\infty P(>z \mid i) = \frac{i-n+1}{n\,\lambda_d}, \quad i \geqq n, \tag{11.36}$$

$$M = \int_0^\infty P(>z) = \sum_{i=n}^{n+m-1} M(i)\,P(i). \tag{11.37}$$

If $\dfrac{\lambda_a}{\lambda_d} < n$, we have for $m \to \infty$, which means that no calls are lost in or transferred from the group, that

$$P(>z) = \sum_{i=0}^{\infty} P(i)\; e^{-zn\lambda_d} \sum_{\nu=0}^{i-n} \frac{(zn\lambda_d)^\nu}{\nu!}, \quad i \geqq n, \tag{11.38}$$

which, combined with (11.32), leads to

$$P(>z) = P(n)\frac{1}{1-\dfrac{\lambda_a}{n\lambda_d}}\, e^{-z(n\lambda_d-\lambda_a)}, \tag{11.39}$$

while the mean waiting time will be

$$M = \frac{P(n)}{n\lambda_d}\,\frac{1}{\left(1-\dfrac{\lambda_a}{n\lambda_d}\right)^2}. \tag{11.40}$$

It follows from (11.33) that $\lim\limits_{m\to\infty} \lambda_{a'} = 0$ so that (11.39) and (11.40) agree with the results found in the case of the pure waiting time problems (8.26) and (8.27).

In the case of the second example where surplus calls are not transferred to a free position, the results will be somewhat simpler since it is no longer necessary to distinguish between "new calls" and "transferred calls".

The assumption can therefore be expressed by (3.1) for new calls, λ_a being replaced by $\lambda_a + \lambda_{a'}$; by (3.2) for terminated calls; by (8.1) for the origination of the waiting time; and by (5.1) for lost calls, n being replaced by $n + m$. The stochastic process $P(i,t \mid j,t_0)$, must thus satisfy the differential equations (3.17) for $i = 0$; (3.18) for $0 < i < n$; (8.16) for $n \leqq i < n + m$; and (11.30) for $i = n + m$, λ_a being everywhere replaced by $\lambda_a + \lambda_{a'}$; and the initial condition.

If the process attains statistic equilibrium we have

$$\sum_{i=0}^{n+m} P(i) = 1. \tag{11.41}$$

The limiting distribution $P(i)$, which is determined by the above mentioned differential equations in connexion with (11.41) and (3.30), will be

$$P(i) = \begin{cases} k_n \dfrac{\left(\dfrac{\lambda_a+\lambda_{a'}}{\lambda_d}\right)^i}{i!}, & 0 \leqq i \leqq n, \\[2ex] P(n)\left(\dfrac{\lambda_a+\lambda_{a'}}{n\lambda_d}\right)^{i-n}, & n < i \leqq n+m, \end{cases} \tag{11.42}$$

where k_n is determined by (11.41).

In order that all groups of an exchange may enter into statistic equilibrium, the handled amount of traffic per group must be equal to the arriving amount of traffic, that is to say,

$$\sum_{i=0}^{n} i\,P(i) + n \sum_{i=n+1}^{n+m} P(i) = \frac{\lambda_a}{\lambda_d} \tag{11.43}$$

or

$$\frac{\lambda_a + \lambda_{a'}}{\lambda_d} \frac{k_n}{k_{n-1}} = \frac{\lambda_a}{\lambda_d}, \tag{11.44}$$

from which $\lambda_{a'}$ can be found.

For $n = 1$ we obtain

$$\frac{\lambda_a + \lambda_{a'}}{\lambda_d} \frac{\sum_{i=0}^{m} \left(\frac{\lambda_a + \lambda_{a'}}{\lambda_d}\right)^i}{\sum_{i=0}^{m+1} \left(\frac{\lambda_a + \lambda_{a'}}{\lambda_d}\right)^i} = \frac{\lambda_a}{\lambda_d}. \tag{11.45}$$

The waiting time distribution does not differ from that of our first example. The results of (11.34)—(11.40) can be used immediately in connexion with (11.42), only that λ_a must be replaced by $\lambda_a + \lambda_{a'}$.

12. Loss and Waiting Time Distributions When the Holding Time Follows More General Distributions.

In some of the waiting time distributions that Erlang investigated, the holding time follows the distribution (1.7). A few results based upon the distribution system (1.15) shall be mentioned in the following.

The Polynomial Process.

The deduction of the binomial process is based upon the one-individual process indicating the probability that the individual will be under observation when the duration of the stay (the holding time) follows the distribution system (1.15) for $f = 1$; if, on the other hand, the holding time follows the general distribution system (1.15), a more general process may be deduced, known as the polynomial process.

First we investigate the probability that a call will be in progress and the state i will be prevailing at the time t, where the state i means that the call will not be terminated until i events have occurred, while the state 0 means that there is no call in progress. The intensities of transition from one state, i, to the next state, $i - 1$, are called λ_i for $i > 0$. The intensity of transition from the state 0 to the state f is called λ_0. The polynomial process for one individual as given by the conditional probability $P(i,t \mid j,t_0)$ can therefore be determined in the usual manner by the initial condition in connexion with the following system of differential equations:

$$P'_t(t,t_0) = AP(t,t_0), \tag{12.1}$$

where

$$A = \left\{ \begin{array}{cccccc} -\lambda_0 & \lambda_1 & 0 \ldots & 0 & 0 \\ 0 & -\lambda_1 & \lambda_2 \ldots & 0 & 0 \\ 0 & 0 & -\lambda_2 \ldots & 0 & 0 \\ . & . & . \ldots & . & . \\ 0 & 0 & 0 \ldots & -\lambda_{f-1} & \lambda_f \\ \lambda_0 & 0 & 0 \ldots & 0 & -\lambda_f \end{array} \right\} \qquad (12.2)$$

The polynomial process can be deduced from this process by the same reasoning as that forming the basis of the deduction of the binomial process for $f = 1$. The limiting distribution $P(\nu_0, \nu_1, \ldots, \nu_f)$ for the polynomial process, where $\sum_{s=0}^{f} \nu_s = N$, will be

$$P(\nu_0, \nu_1, \ldots, \nu_f) = \frac{N!}{\nu_0!\,\nu_1! \ldots \nu_f!} \left(\frac{1}{\lambda_0}\right)^{\nu_0} \left(\frac{1}{\lambda_1}\right)^{\nu_1} \ldots \left(\frac{1}{\lambda_f}\right)^{\nu_f} \frac{1}{\left(\sum_{s=0}^{f} \frac{1}{\lambda_s}\right)^N} \qquad (12.3)$$

which indicates the probability for $\nu_0, \nu_1, \ldots, \nu_f$ calls in the states 0, 1, $\ldots, f$. The distribution (12.3) shows *inter alia* that the probability for ν_0 free switches among N switches is independent of the distribution of the holding time since it is of no importance whether the holding time is distributed in one or the other of the distributions comprised in the system (1.15), if only

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_a} \quad \text{and} \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_f} = \frac{1}{\lambda_d}, \qquad (12.4)$$

which means that the limiting distribution of the number of free switches among altogether N switches depends exclusively on the amount of traffic handled over the group of switches concerned.

Waiting Time Distribution, One Switch.

We will consider a case where we have one switch in a telephone exchange with waiting time arrangement. When we say that the state i is prevailing at the arrival of a call, it means that i events must occur in a certain order before the call is given access to the switch; thus, when $i = sf + \nu$ there will be s previously arrived calls waiting, and the call that is in progress will not be terminated until ν events have occurred. The probability that the state i will be prevailing at the arrival of a call at the time t, when the state j was prevailing at the time t_0 previous to t, is denoted by $P(i,t \mid j,t_0)$ and determined by a system of differential equations having the matrix

$$A = \{ a_{ij} \} \qquad (12.5)$$

where

$$
\begin{aligned}
&a_{fs+\nu-1,\, fs+\nu} = \lambda_\nu, && \nu = 1, \ldots, f,\ s = 0, 1, \ldots, \\
&a_{0,0} = -\lambda_0, && \\
&a_{fs+\nu,\, fs+\nu} = -\lambda_\nu - \lambda_0, && \nu = 1, \ldots, f,\ s = 0, 1, \ldots, \\
&a_{j+f,\, j} = \lambda_0, && j = 0, 1, \ldots,
\end{aligned}
$$

while all other elements are zero.

In the case treated by Erlang we have $\lambda_0 = \lambda_a$ and $\lambda_\nu = f\lambda_d$ for $\nu = 1, \ldots, f$, which means that the limiting distribution $P(i)$ for statistic equilibrium must satisfy the difference equations

$$P(i+1) = q(P(i) + P(i-1) + \cdots + P(i-f+1)), \qquad i \geqq f, \quad (12.6a)$$

and

$$P(i+1) = q(P(i) + P(i-1) + \cdots + P(0)), \qquad 0 \leqq i < f, \quad (12.6b)$$

where $q = \dfrac{\lambda_a}{f\lambda_d} = \dfrac{1}{f}\, a$, and a indicates the amount of traffic offered to the switch which, in this case, is equal to the amount of traffic handled over the switch.

The solutions of the difference equations (12.6a) may be written

$$P(i) = c_1 r_1^i + \cdots + c_f r_f^i \quad (12.7)$$

where $r_1, r_2, \ldots, r_f$ are the f different roots of the polynomial

$$r^f - q(r^{f-1} + r^{f-2} + \cdots + 1) = 0. \quad (12.8)$$

As $P(0) = 1 - a$, it follows from (12.6b) that

$$P(i) = \sum_{\mu=0}^{i-1} \binom{i-1}{\mu} q^{i-\mu} P(0) = q(1-a)\,(1+q)^{i-1}, \qquad 1 \leqq i \leqq f-1. \quad (12.9)$$

The arbitrary constants $c_1, \ldots, c_f$ in (12.7) are therefore given such values that (12.7) will also satisfy the other difference equations (12.6b); this means that

$$\begin{Bmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_f \\ \cdot & \cdot & \ldots & \cdot \\ r_1^{f-1} & r_2^{f-1} & \ldots & r_f^{f-1} \end{Bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ c_f \end{Bmatrix} = \begin{Bmatrix} P(0) \\ P(i) \\ \cdot \\ P(f-1) \end{Bmatrix}, \quad (12.10)$$

which has one and only one solution, as $r_\mu \neq r_\nu$ for $\mu \neq \nu$.

The probability $P(>t,f)$ that a call will have to wait more than the time t will then, by (12.7) and (1.4), be

$$P(>t,f) = \sum_{i=1}^{\infty} P(i)\, P(>t,i) \tag{12.11}$$

$$= \sum_{i=1}^{\infty} P(i) \int_{f\lambda_d t}^{\infty} \frac{y^{i-1}}{(i-1)!} e^{-y}\, dy$$

$$= \int_{f\lambda_d t}^{\infty} dy \sum_{\nu=1}^{f} c_\nu r_\nu e^{(r_\nu - 1)\, y}$$

$$= \sum_{\nu=1}^{f} \frac{c_\nu r_\nu}{1 - r_\nu} e^{(r_\nu - 1)\, f\lambda_d t}.$$

The mean waiting time M will be

$$M = \int_0^{\infty} P(>t,f)\, dt = \frac{1}{f\lambda_d} \sum_{\nu=1}^{f} \frac{c_\nu r_\nu}{(1 - r_\nu)^2} \tag{12.12}$$

which may be written

$$M = \frac{1}{f\lambda_d (1-a)^2} \sum_{\nu=1}^{f} c_\nu r_\nu \prod_{s \neq \nu} (1 - r_s)^2; \tag{12.13}$$

using (12.8) and (12.10) we find that

$$M = \frac{f+1}{2f} \; \frac{a}{1-a} \; \frac{1}{\lambda_d}. \tag{12.14}$$

Letting $f \to \infty$ we obtain the result found in (10.58) for $n = 1$.

13. *Statistic Equilibrium. — Ergodic Theory.*

Most of the results obtained in the foregoing by application of the principle of statistic equilibrium can be recapitulated in a more general form; a recapitulation, especially of the processes where the holding time (the stay) follows the distribution (1.7), is given below. These processes satisfy the following assumptions:

The probability that an arrival will occur during a given time interval when ν individuals are present at the origination of the interval is asymptotically proportional to the length of the interval, with a factor of proportionality $\beta_{\nu,1}$ that is independent of time; $\nu = 0, 1, 2, \ldots, N-1$. (13.1)

The probability that a departure will occur during a given time interval when ν individuals are present at the origination of the interval is asymptotically proportional to the length of the interval, with a factor of proportionality α_ν that is independent of time; $\nu = 1, 2, \ldots, N$. (13.2)

In (13.1) and (13.2) N may be finite or infinite.

In order to determine the process $P(i,t \mid j,t_0)$, we investigate the value of the process at the time $t + \Delta t$ using the combinations of numbers of individuals present at the time t and numbers of arrivals and departures occurring during the time interval Δt that result in the presence of i individuals at the time $t + \Delta t$. After a limit passage the following differential equation is obtained:

$$P_t'(i,t) = \beta_{i-1,1} P(i-1,t) - (\beta_{i,1} + \alpha_i) P(i,t) + \alpha_{i+1} P(i+1,t), \quad 0 \leqq i \leqq N, \tag{13.3}$$

where $\beta_{-1,1} = \alpha_0 = \beta_{N,1} = \alpha_{N+1} = 0$.

The process must further satisfy the condition

$$P(i,t_0 \mid j,t_0) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{13.4}$$

Using the matrix notation mentioned in Chapter 4, we may write (13.3) and (13.4) in the form

$$P_t'(t,t_0) = AP(t,t_0) \tag{13.5}$$

$$P(t_0,t_0) = E \tag{13.6}$$

where the elements of the matrix $A = \{a_{\mu\nu}\}$ are

$$\begin{aligned} a_{\nu-1,1} &= \alpha_\nu, & \nu &= 1, 2, \ldots, N \\ a_{\nu,\nu} &= -(\beta_{\nu,1} + \alpha_\nu), & \nu &= 0, 1, 2, \ldots, N \\ a_{\nu+1,\nu} &= \beta_{\nu,1} & \nu &= 0, 1, 2, \ldots, N-1 \end{aligned} \tag{13.7}$$

and all other elements are zero. N can be finite as well as infinite.

A matrix $A_{n+m,\,n+m} = \{a_{\mu\nu}\}$ may, *e. g.*, be composed of the following 4 matrices:

$$A_{nn} = \{a_{\mu\nu}\}, \quad \begin{aligned} \mu &= 1, \ldots, n, \\ \nu &= 1, \ldots, n, \end{aligned} \tag{13.8}$$

$$A_{nm} = \{a_{\mu\nu}\}, \quad \begin{aligned} \mu &= 1, \ldots, n, \\ \nu &= n+1, \ldots, n+m, \end{aligned} \tag{13.9}$$

$$A_{mn} = \{a_{\mu\nu}\}, \quad \begin{aligned} \mu &= n+1, \ldots, n+m, \\ \nu &= 1, \ldots, n, \end{aligned} \tag{13.10}$$

$$A_{mm} = \{a_{\mu\nu}\}, \quad \begin{aligned} \mu &= n+1, \ldots, n+m, \\ \nu &= n+1, \ldots, n+m, \end{aligned} \tag{13.11}$$

written

$$A_{n+m,\, n+m} = \begin{Bmatrix} A_{nn} & A_{nm} \\ A_{mn} & A_{mm} \end{Bmatrix}. \tag{13.12}$$

This means, obviously, that the original matrix has been divided into 4 parts by a vertical and a horizontal line. This partition may be continued.

If the matrix A by some arbitrary numbering of the states can be divided so that all elements of the 2 matrices A_{mn} and A_{nm} are zero and the elements of the other 2 matrices are different from zero, then the states $1, \ldots, n+m$ are *"absolutely divisible in isolated groups"* (*Class I*), using the terminology of *v. Mises*[1]).

If only one of the matrices A_{mn} and A_{nm} can become a zero matrix by partition, the other 3 matrices having elements different from zero, then the states $1, \ldots, n$ are called *a group without any probabilities for departure, or without any probabilities for arrival, respectively* (*Class II*).

If it is not possible to perform a partition turning one of the matrices into a zero matrix, then the states $1, \ldots, n+m$ are *"absolutely indivisible"* (*Class III*).

The processes given by (13.5) belong in *Class I* if *e.g.* there exist pairs of elements $\alpha_{\nu+1} = \beta_\nu = 0$; in *Class II*, if some elements $\alpha_\nu = 0$ and all $\beta_\nu \neq 0$, or if some elements $\beta_\nu = 0$ and all elements $\alpha_\nu \neq 0$; and in *Class III*, if all elements $\alpha_\nu \neq 0$ and $\beta_\nu \neq 0$.

Such stochastic processes as may enter into statistic equilibrium are especially interesting.

The concept of *statistic equilibrium* is an extension of the so-called ergodic principle which was developed in connexion with the application of statistics to physical problems. *Boltzmann* and *Maxwell* tried to solve some such problems by introducing *the ergodic principle.*[2])

Kolmogoroff gives the following definition[3]):

A stochastically definite process is said to follow the ergodic principle (enter into statistic equilibrium) when

$$\lim_{t \to 0} [P(E, t \mid j, t_0) - P(E, t \mid k, t_0)] = 0 \tag{13.13}$$

is valid for arbitrary values of t_0, j, k, and E, where E denotes a group of possible states.

If the matrix A is constant and the stochastic process, accordingly, is homogeneous with respect to time, this can obviously be written:—

[1]) *Ricard v. Mises* (1931).

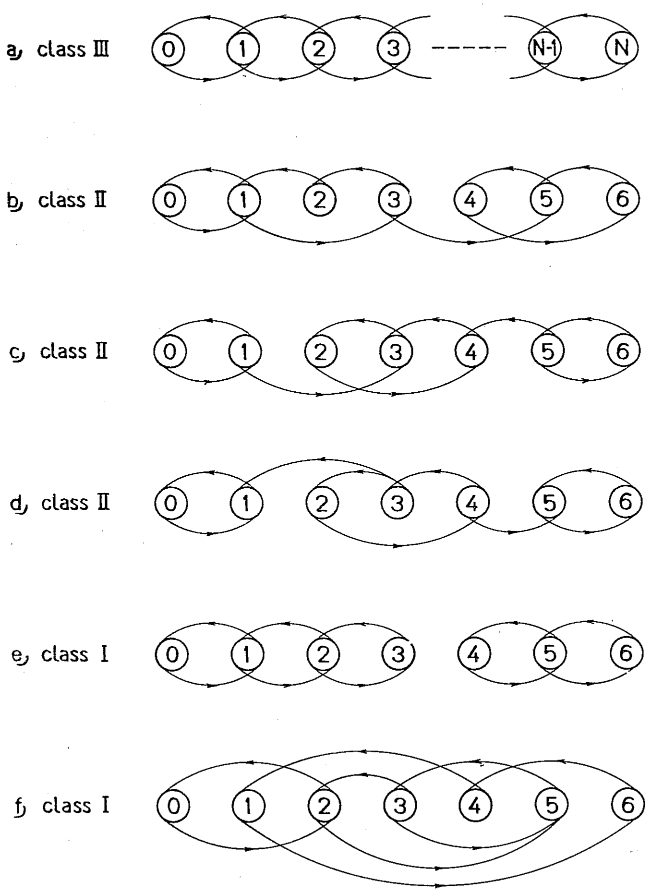
[2]) A detailed account of these problems is given by *B. & T. Ehrenfest* (1909).

[3]) *Kolmogoroff* (1931).

A stochastically definite process can enter into statistic equilibrium when

$$\lim_{\sigma \to \infty} [P(E, \sigma \mid j, 0) - P(E, \sigma \mid k, 0)] = 0 \tag{13.14}$$

is valid for arbitrary values of j, k, and E.



a, b, and c, only one group without departures (rank = N)
d, e, and f, two groups without departures (rank < N)

Fig. 13.1.

(13.14) will, for processes with a finite number of states ($N < \infty$), be satisfied if the rank of the matrix A is N. The question of whether a process is of the rank N can be settled, in concrete problems, by means of a simple drawing in which the possible states are numbered 0, 1, 2, ..., N, and the possible direct jumps from state to state are indicated by arrows, as shown in fig. 13.1 where jumps to "higher" and "lower" states are represented by downward and upward arcs, respectively.

It will appear directly from this whether the chain of states is continuous or divided into two or more mutually independent parts. If there are two or more groups of states without possibility of departures, the rank will be less than N; otherwise it will be equal to N. In the case of the processes determined by (13.5) and (13.6) this can be expressed as follows[1]):

In order that a process (13.5) with a limited number of states may enter into statistic equilibrium, it is a necessary and sufficient condition that there be no such set of numbers (j, k) that

$$\beta_j = \alpha_k = 0, \quad j < k. \tag{13.15}$$

The states $0, 1, 2, \ldots, N$ can be divided into a series of groups $S_1, \ldots S_r$ in such a way that the states within any one group directly or indirectly are mutually connected. Let S_1 and S_2 be two groups without possibility of departures, and let i and j belong to S_1 and k to S_2; we then have $P(k, t \mid j, t_0) \equiv 0$, whereas $\lim\limits_{t \to \infty} P(i, t \mid j, t_0)$ is positive. (13.15) is thus a necessary condition; but it is also sufficient since there is, for any group S without possibility of departures, a single characteristic root $r = 0$ in the secular equation corresponding to A, whereas the real part of the other roots is negative[2]).

The limiting distributions for these processes are determined by the equations

$$O = AP \tag{13.16}$$

where

$$P = \begin{Bmatrix} P(0) \\ \vdots \\ P(N) \end{Bmatrix}$$

and

$$\sum_{i=0}^{N} P(i) = 1. \tag{13.17}$$

Hence it follows that

$$P(i) = P(0)\,\frac{\beta_0\,\beta_1 \cdots \beta_{i-1}}{\alpha_1\,\alpha_2 \cdots \alpha_i}, \quad i = 1, 2, \ldots, N, \tag{13.18}$$

where $P(0)$ is determined by (13.17). This distribution comprises all those processes treated in the foregoing that have a finite number of states $(N + 1)$ and where the holding time follows the distribution (1.7).

[1]) I am extremely grateful to Prof. *G. Elfving* for valuable information about this proof. *A. J.*

[2]) *Romanovsky* (1909). *v. Mises* (1931). *B. Hostinsky* (1931). *Alexander Rajchman* (1930).

If the above-mentioned processes with an infinite number of states are to be able to enter into statistic equilibrium, it is furthermore necessary that $P(E, t \mid j, t_0)$ be uniformly convergent in E for $t \to \infty$, and that

$$\sum_i P(i,t \mid j,t_0) = 1, \tag{13.19}$$

while it is a sufficient condition that

$$\alpha_\nu + \beta_\nu < k_1 \nu, \quad \nu = 1, 2, \ldots$$

If $P(E,t \mid j,t_0)$ converges uniformly in E for $t \to \infty$ and (13.19) is satisfied, we have

$$\lim_{t \to \infty} P(i, t \mid j, t_0) = P(i) \tag{13.20}$$

and

$$\sum_i P(i) = 1,$$

where $P(i)$ is determined by (13.18) for N infinite.

It is thus a necessary condition for statistic equilibrium in the case of the processes here treated that

$$S_n = \sum_{i=1}^{n} \frac{\beta_0 \beta_1 \cdots \beta_{i-1}}{\alpha_1 \alpha_2 \cdots \alpha_i} \tag{13.21}$$

converges for $n \to \infty$.

(13.21) will always be satisfied if

$$\frac{\beta_{\nu-1}}{\alpha_\nu} \leqq k_2 < 1 \quad \text{for} \quad \nu > M. \tag{13.22}$$

Thus, (13.18) contains all the problems treated by Erlang where the distribution of the holding time is (1.7); it is also valid without this restriction in the case of the so-called loss problems[1]).

In Chapter 12 we investigated a process whose matrix $A = \{a_{ij}\}$ had the elements

$$\begin{aligned} a_{j-1,j} &= \alpha_j, & j &= 1, 2, \ldots, N, \\ a_{jj} &= \alpha_j - \beta_{jf}, & j &= 0, 1, \ldots, N, \\ a_{j+f,j} &= \beta_{jf}, & j &= 0, 1, \ldots, N-1, \end{aligned} \tag{13.23}$$

all other elements being zero, and with N finite as well as infinite.

For N finite, this process will — like those mentioned above — enter into statistic equilibrium when its states cannot be divided into two groups without possibility of departures and the elements are bounded.

[1]) See Chapter 10.

In the waiting time process with an unlimited number of states, treated by Erlang, the process $P(E, t \mid j, t_0)$ will not converge uniformly in E for $t \longrightarrow \infty$ unless

$$\frac{\lambda_a}{f\lambda_d} < \frac{1}{f}, \tag{13.24}$$

which means that the amount of traffic handled over the circuit must be less than 1 erlang.

The mean value (with respect to time) of the stochastic processes considered here is especially significant for their applications.

If

$$M(i, t_0 + T \mid j, t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} P(i, t \mid j, t_0)\, dt, \tag{13.25}$$

we have

$$M(i, t_0 + T \mid j, t_0) - P(i) = \frac{1}{T} \int_{t_0}^{t_0+T} (P(i, t \mid j, t_0) - P(i))\, dt$$

$$= \frac{1}{T} \int_{t_0}^{t(\epsilon)} (P(i, t \mid j, t_0) - P(i))\, dt + \frac{1}{T} \int_{t(\epsilon)}^{t_0+T} (P(i, t \mid j, t_0) - P(i))\, dt; \tag{13.26}$$

if $P(i, t \mid j, t_0)$ converges uniformly in i and j, there will exist a $t(\epsilon)$ corresponding to ϵ for which it is valid, for all i and j, that

$$\left| P(i, t \mid j, t_0) - P(i) \right| < \epsilon \quad \text{for} \quad t \geqq t(\epsilon), \tag{13.27}$$

which means that

$$\left| M(i, t_0 + T \mid j, t_0) - P(i) \right| \leqq \frac{t(\epsilon) - t_0}{T} + \frac{1}{T}(t_0 + T - t(\epsilon))\, \epsilon. \tag{13.28}$$

Hence it follows that there, for any ϵ_1, exists a $T(\epsilon_1)$ such that

$$\left| M(i, t_0 + T \mid j, t_0) - P(i) \right| < \epsilon_1 \quad \text{for} \quad T \geqq T(\epsilon_1) \tag{13.29}$$

for all i and j, which means that

$$\lim_{T \longrightarrow \infty} M(i, t_0 + T \mid j, t_0) = P(i) \tag{13.30}$$

when the stochastic process concerned can enter into statistic equilibrium.

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A SURVEY OF A. K. ERLANG'S MATHEMATICAL WORKS

By E. BROCKMEYER.

A. Works Concerning the Theory of Probability.

The greatest, and by far the most important, part of Erlang's production comprises his works concerning the application of the theory of probability to problems of telephone traffic. The investigation of these problems constituted an essential part of his activities throughout the 20 years he spent as a scientific collaborator of the Copenhagen Telephone Company.

Characteristic of Erlang's achievements within this field are his endeavours to deduce as much as possible from a single basic principle. In the case of these problems he found this basic principle in the assumption of statistic equilibrium, a concept which was known, it is true, from other domains; it was Erlang's works, however, that disclosed the wealth of possibilities contained in this principle with regard to the theory of telephone traffic. The mathematically exact methods of solving problems of loss and waiting times, which Erlang developed by his employment of the principle of statistic equilibrium, are of fundamental importance in the theory of telephone traffic.

On his results within this domain, Erlang has published the works mentioned below under Nos. 1—5[1]. As most of the contents of these publications are treated in detail in *Arne Jensen's* paper "An Elucidation of Erlang's Statistical Works Through the Theory of Stochastic Processes" (p. 23), a brief survey will suffice here:

1. *The Theory of Probabilities and Telephone Conversations*, p. 131.

   First published in Danish:
   Sandsynlighedsregning og Telefonsamtaler.
   Nyt Tidsskrift for Matematik B, vol. 20, 1909, p. 33.
   Later published in French:
   Calcul des probabilités et conversations téléphoniques.
   Revue général de l'Electricité, vol. 18, 1925, p. 305.

[1]) The numbers prefixed in this survey to the titles of Erlang's reprinted works correspond to the numbers of the reprints in the present book, to which also the suffixed page numbers have reference.

In this, his first published work on the theory of telephone traffic, Erlang proves in Section 1 that the number of calls originating during a given time interval, assuming random origination of the calls, follows the Poisson law of distribution. Some properties of this law of distribution are mentioned in Section 2. Section 3 deals with the simplest case of the problems of waiting time for calls into a group of circuits when the holding times are constant, *viz.* the case where the group consists of one circuit only. The problem is here formulated so as to concern calls to an operator serving a manual position. In the original Danish edition Erlang then gave a brief treatment, in Section 4, of the corresponding problem for a group consisting of several circuits, so formulated as to concern a distribution system with a group of cooperating operators. An error had slipped into the treatment of this problem, however, and as the problem had been given a new and better treatment in the work from 1917 mentioned below as No. 2, Erlang omitted this section in the French edition from 1925 and inserted instead a new Section 4 containing some supplementary remarks and tables to Section 3. The present reprint conforms to the French edition.

2. *Solution of some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges*, p. 138.
   First published in Danish:
   Løsning af nogle Problemer fra Sandsynlighedsregningen af Betydning for de automatiske Telefoncentraler.
   Elektroteknikeren, vol. 13, 1917, p. 5.
   Also published in English, the numerical tables not included, under the above-cited title in The Post Office Electrical Engineers' Journal, vol. 10, 1918, p. 189.
   Furthermore published in German and French:
   Lösung einiger Probleme der Wahrscheinlichkeitsrechnung von Bedeutung für die selbsttätigen Fernsprechämter.
   Elektrotechnische Zeitschrift (E. T. Z.), vol. 39, 1918, p. 504.
   Solutions de quelques problèmes de la théorie des probabilités présentant de l'importance pour lex bureaux téléphoniques automatiques.
   Annales des Postes, Télégraphes et Téléphones, vol. 11, 1922, pag. 800.

This work must be regarded as Erlang's most important publication. In Section 1—7, the loss problem for a simple group of circuits is dealt with on the basis of the principle of statistic equilibrium, and Erlang here sets forth and proves his famous B-formula for the loss which is of fundamental significance to the theory of telephone traffic. It should be noted, however, that the proof which Erlang gives in Sections 2—5, is correct only in the case of exponential distribution of the holding times (which presupposition Erlang mentions in Section 6), but not in the case of constant holding times. (For further information, see pp. 33-38). In connexion with the *B*-formula are given two tables of numerical values, of which Table 1 indicates the loss, *B*, for fixed values of the number of circuits, *x*, and the traffic intensity, *y*, whereas Table 2 indicates the traffic intensity, *y*, corresponding to fixed values of the number of circuits, *x*, and the loss, *B*[1]). Besides, Erlang's interconnexion formula (later published in the work mentioned below under No. 4) is briefly mentioned in Section 7, and some numerical values to this formula are given in Table 3.

Section 8 deals with the waiting time problem for calls originated to a group of circuits when the holding times are constant, *i. e.* the general case of the problem, the simplest case of which, *viz.* that of only one circuit, was treated by Erlang in his work referred to as No. 1 in the above. A collocation of the formulae for 1, 2, and 3 circuits, with appurtenant numerical tables (Tables 4—6) is given, but Erlang does not prove these formulae here. The later published work, referred to as No. 3 below, contains a more detailed exposition of this problem.

Section 9 deals with the waiting time problem for calls originated to a group of circuits when the holding times are distributed exponentially. The exposition is very brief, and Erlang sets forth, without proofs, the general formulae for the solution of this problem, valid for any number of circuits; corresponding numerical values are given in Table 7. These formulae are just as fundamentally significant to the theory of waiting time as the B-formula is to the theory of loss.

In Section 10, some approximative formulae are mentioned, including an approximative formula expressing the loss by means of the Poisson series, with numerical values given in Table 8. Finally, some supplementary remarks are made in Sections 11—12.

3. *Telephone Waiting Times*, p. 156.

First published in Danish:
Telefon-Ventetider. Et Stykke Sandsynlighedsregning.
Matematisk Tidsskrift B, vol. 31, 1920, p. 25.
Later published in French:
Calcul des probabilités et conversations téléphoniques.
Revue générale de l'Electricité, vol. 20, 1926, p. 270.

[1]) The reader will find a more comprehensive table, corresponding to Erlang's Table 2, on page 268. A detailed six-figure table corresponding to Erlang's Table 1 has also been computed; it is not included in the present book, however, as a similar table has recently been published by *Conny Palm* in "Tables of Telephone Traffic Formulae" Nr. 1, Stockholm, 1947.

This paper is Erlang's principal work on waiting times, assuming constant holding times. While Erlang in No. 2 above only stated the results for 1, 2, and 3 circuits without proofs, he treats the problem thoroughly here. Erlang illustrates his method of finding a general solution of the problem by deducing in full the formulae for 1 and 2 circuits, in Sections 2—6 and 7—11, respectively. This work is one of the least perspicuous of Erlang's papers, however, owing to his peculiar mode of expression. A more easily comprehensible treatment of the problem has later been published by *C. D. Crommelin*[1]) who has given Erlang's solution a mathematically more elegant form.

4. *On the Application of the Theory of Probabilities in Telephone Administration*, p. 172.

   First published in Danish:
   Sandsynlighedsregningens Anvendelse i Telefondrift.
   Første nordiske Elektroteknikermøde i København 1920 (H. C. Ørstedmødet), Copenhagen, 1922, p. 149; reprinted in: Elektroteknikeren, vol. 19, 1923, p. 99.
   Later published in French:
   Application du calcul des probabilités en téléphonie.
   Annales des Postes, Télégraphes et Téléphones, vol. 14, 1925, p. 617.

This work forms the basis of a lecture read by Erlang before the First Scandinavian Congress of Electrotechnicians in Copenhagen, 1920; it contains, partly, a survey — without proofs — of Erlang's earlier results from the publications mentioned under Nos. 1—3, and partly, a statement of some new results, also without proofs.

After an introduction containing some historical remarks, the Poisson distribution of calls is mentioned in Section 2a. Section 2b deals with various hypotheses for the holding time, and Table 1 shows the results of a statistic of holding times, made at the Main Exchange of Copenhagen in 1916, indicating that the holding time follows, with sufficient approximation, the law of exponential distribution. The *B*-Formula of loss is mentioned in Section 3a, and the results of some artificial traffic records are, in this connexion, shown in Table 4.

In Sections 3b and 3c Erlang states his solution of two new problems, *viz.*, in Section 3b, the loss problem in the case of binomial distribution of the calls, the resulting formulae being compiled in Table 3, and, in Section 3c, the loss problem in the case of systems with "grading and inter-

[1]) *C. D. Crommelin:* Delay Probability Formulae when the Holding Times are Constant. Post Office Electrical Engineers Journal, vol. 25, April 1932, p. 41, and vol. 26, January, 1934, p. 266.

connecting"; Erlang's interconnexion-formula is given in Table 3 with appurtenant numerical values in Table 5. Since Erlang has not given any proof of this important formula, and since it is not mentioned in Arne Jensen's paper, I give a proof of it in Appendix 1, p. 113.

In Section 4a the problem of waiting time when the holding times are constant is dealt with; Table 6 is a collocation of the formulae for 1, 2, and 3 circuits, with curves of the corresponding numerical values given in Tables 7—9. More curves and numerical tables for the waiting time theory for constant holding times are given in Tables 13—17. The latter tables are not to be found in the Danish edition, but are added in the French translation. In Section 4b the waiting time problem for exponentially distributed holding times is briefly mentioned, and in Section 4c with Table 10 various other hypotheses for the distribution of the holding times.

In Section 5a, a comparison of busy-signal systems and waiting-time systems is made; and the theory of waiting time in manual positions, assuming operators' team work or automatic distribution of calls, respectively, is treated in Sections 5b and 5c, with appurtenant collections of formulae in Tables 11 and 12.

5. *Some Applications of the Method of Statistical Equilibrium in the Theory of Probabilities*, p. 201.

Den sjette skandinaviske Matematikerkongres i København 1925, Copenhagen, 1926, p. 157.

Later published in French:

Quelques applications de la méthode de l'équilibre statistique dans la théorie des probabilités.

Annales des Postes, Télégraphes et Téléphones, vol. 17, 1928, p. 743.

This work forms the basis of a lecture given by Erlang at the Sixth Scandinavian Congress of Mathematicians in Copenhagen, 1925; it contains a survey of the most important of Erlang's earlier results with respect to distribution of calls, loss problems, and waiting time problems, from the publications mentioned above under Nos. 1—4.

In this paper, the significance of the principle of statistic equilibrium to these problems is strongly emphasized and a number of new figures are added, in Tables 1—4, in order to illustrate the application of this principle; also, some references are given to its application in other domains than the theory of telephone traffic.

Erlang's interconnexion formula — published in the work here referred to as No. 4 — is, besides, mentioned in two letters from Erlang to *G. F.*

O'Dell; an extract of the letters has been published in the form of an appendix to a treatise by O'Dell[1]).

The appendix comprises four tables, of which Tables 1 and 3 containing, respectively, the interconnexion formulae and numerical values of the loss as calculated by means of this formula, are reproductions of Tables 3 and 5, respectively, in the above mentioned work No. 4, whereas Table 2 contains numerical values of the quantities N and T involved in the formula, and Table 4 gives the logarithms of certain binomial coefficients.

Erlang has, in addition to the published papers on the theory of telephone traffic mentioned above, in the course of his employment at the Copenhagen Telephone Company written various notes about problems related to the subject. Most of these, however, are not of general interest as they treat concrete problems at hand by means of the principles and methods of calculation given in Erlang's published works; they were not written for publication, but only for use inside the walls of the Telephone Company.

The following work makes an exception, though; it treats of a new principle for the calculation of the number of circuits, and it is written in such a form as to render it well fitted for publication:

6. *On the Rational Determination of the Number of Circuits.* p. 216.

Written in 1924. First published in the present book.

The usual practice when calculating the number of lines or switches has hitherto been that of prescribing certain values for the loss or the waiting time, *e. g.*, in the case of busy-signal systems, the loss $B = 0.002$.

This principle is, however, open to weighty objections. A group consisting of few circuits can carry only a very small amount of traffic per circuit, compared with a large group of circuits calculated to give the same loss; the small group will consequently be much too expensive, considering its traffic capacity. For the sake of economy larger values of the loss are therefore often tolerated for small groups, these values being fixed in a rather arbitrary manner not based upon any rational principle.

Besides, the calculation of the number of circuits for a fixed value of loss or waiting time causes large groups to be considerably more sensitive to congestion than smaller groups, so that a certain increase of traffic beyond the normal traffic involves a considerably greater increase of the

[1]) *G. F. O'Dell:* The Influence of Traffic on Automatic Exchange Design. The Institution of Post Office Electrical Engineers, Publication No. 85, London, 1920, Appendix 1, pp. 35—47.

loss or the waiting time in the large groups than in the smaller ones. Then, by way of making up for this circumstance, an extra prescription is often added, to the effect that the loss or the waiting time must not increase beyond a certain limit for a given excess of traffic. Thus, in a frequently used prescription the condition that the loss must not exceed $B = 0.002$ for normal traffic is supplemented with the condition that the loss must not exceed $B = 0.01$ for a 10 per cent. increase of the traffic. Since the two corresponding computation curves intersect when the number of circuits $= 70$, it is obvious that the curve corresponding to $B = 0.002$ is to be followed for groups of less than 70 circuits, whereas, for groups of more than 70 circuits, the other curve must be followed.

It stands to reason that such calculating principles cannot be regarded as a satisfactory solution of the problem. A rational method of calculation should follow a prescription that is uniformly applicable to large as well as to small groups.

Such a rational principle for the calculation of the number of circuits has been given by *K. Moe*, Engineer-in-Chief to the Copenhagen Telephone Company. Moe's principle is based on the following reasoning: The improvement of the traffic conditions gained by adding one new circuit to a group of circuits consists in a certain reduction of the number of lost calls or waiting time units per unit of time. Accordingly, in order that the calculation of the number of circuits in the group can be called economically rational, the "improvement" — *i. e.* the decrease of the number of lost calls or waiting time units per unit of time obtained by adding one circuit to the group — must be the same whether the number of circuits in the group is large or small. The value of the "improvement" to be prescribed for a certain group of circuits must be fixed with due regard to the expense per circuit.

When the number of circuits is calculated on the basis of a fixed value of the "improvement", the loss — or the waiting time, respectively — will decrease evenly as the number of circuits increases. By the use of this principle, the disadvantages of the calculation based upon fixed values of loss or waiting time are avoided without extra prescriptions.

Only two publications dealing with the Moe principle have as yet appeared, a very brief mention in 1931[1]) and a somewhat more detailed representation in 1940[2]).

In his work from 1924, Erlang treats the calculation of simple groups

[1]) "Kjøbenhavns Telefon 1881—1931", Copenhagen, 1931, p. 47; also published in English: "The Development of Telephonic Communication in Copenhagen 1881—1931", Ingeniørvidenskabelige Skrifter A, no. 32, Copenhagen, 1932, p. 142.

[2]) Section 12 of: *E. Brockmeyer:* »Grundtræk af Sandsynlighedsregningens Anvendelse i Telefonien«, Ingeniøren, 1940, p. E21.

of circuits in compliance with this principle and deduces the "improvement" formulae for busy-signal systems (Formula 2) as well as for waiting time systems (Formula 6), supplemented with two tables of numerical values of the "improvement". In connexion with the two "improvement" formulae Erlang further gives two approximative formulae expressing the "improvement" by means of the Gaussian Normal Function (Formulae 3 and 7, respectively); since Erlang has not given any proof of these approximative formulae, I give a deduction of the formulae in Appendix 2, p. 120.

Erlang also took a great interest in the application of the theory of probability outside the field of telephone traffic, such as its application to various physical problems, genetic problems, and statistics of population. He wrote, however, only one paper on these subjects, *viz.*, the following, which treats of a fundamental problem in theoretic physics, Maxwell's Law:

7. *A Proof of Maxwell's Law, the Principal Proposition in the Kinetic Theory of Gases*, p. 222.

   First published in Danish:
   Et Bevis for Maxwells Lov, Hovedsætningen i den kinetiske Luftteori.
   Fysisk Tidsskrift, vol. 23, 1925, p. 40.
   Later published in French:
   Démonstrations de la loi de Maxwell, proposition fondamentale de la théorie des gaz.
   La Vie Technique et Industrielle, vol. 8, 1926, p. 72.

In this work Erlang gives, by means of the principle of statistic equilibrium, an elementary proof of Maxwell's law of distribution for the velocity of gas molecules. Erlang's predilection for a brief and concise style and his faculty of pouncing upon the essential points of a complicated problem appear clearly from this work. After a short historical introduction concerning the application of the theory of probability to physical problems, Erlang formulates the necessary assumptions as to the nature of the gas molecules and the effects of their collisions. Erlang has here reduced the necessary assumptions to a minimum; basing his proof on the very simplest assumptions of mechanics, he does not even use the fundamental theorem of energy. Starting from these minimal assumptions, Erlang then proceeds to give a quite elementary proof of Maxwell's Law, showing by a simple geometrical consideration that the Maxwellian law of distribution satisfies the conditions for a state of statistic equilibrium.

B. Works Concerning Mathematical Tables.

A subject that interested Erlang very much was the calculation and arrangement of numerical tables of mathematical functions, and he had an uncommonly thorough knowledge of the history of mathematical tables from ancient times right up to the present.

Erlang has set forth a new principle for the calculation of certain forms of mathematical tables, especially tables of logarithms, and published the following works about it:

Om Indretningen og Beregningen af fircifrede Logaritmetabeller. (*On the Arrangement and Calculation of Four-Figure Tables of Logarithms*). Nyt Tidsskrift for Matematik B, vol. 21, 1910, p. 55.

A somewhat enlarged version of this work was later published in English:

8. *How to Reduce to a Minimum the Mean Error of Tables*, p. 227.
   The Napier Tercentenary Memorial Volume, Royal Society of Edinburgh, 1915, p. 345.

This English version has been reprinted in the present book. Erlang herein mentions two principal types of tables: Type A, comprising tables adapted for ordinary linear interpolation, and Type B, comprising tables provided with a special auxiliary table for interpolation of tenth parts for each horizontal row in the main table. Erlang then shows how the greatest possible average exactness can be secured by means of the "method of least squares" — well-known from the theory of errors — according to which the table values are to be calculated so that the sum of the squares of the errors of all the values obtainable from the main table and the interpolation table is a minimum.

Capt. *N. E. Lomholt* has written a review of the above-cited Danish paper in Nyt Tidsskrift for Matematik B, vol. 22, 1911, p. 8, in which he makes some critical comments on Erlang's method.

Erlang published, on p. 10 in the same volume, a reply containing some supplementary remarks among which may be mentioned a more explicit statement of reasons for the application of the "method of least squares" to the calculation of tables of logarithms; this remark is reprinted as a foot-note on p. 229.

Erlang has applied his method, as described in these works, in practice to the calculation of the two tabular works mentioned below:

Fircifrede Logaritmetavler og andre Regnetavler til Brug ved Undervisning og i Praksis. (*Four-Figure Tables of Logarithms and other Mathematical Tables for Instruction and Practical Applications*).
Copenhagen, 1911, and later editions.

These tables are available in 3 different editions: A, B, and C, of which the most comprehensive, C, contains four-figure tables of common logarithms and antilogarithms, table of squares, compound interest tables, tables of trigonometric functions and their logarithms, table of natural logarithms, and some supplementary tables. Owing to their exactness and practical arrangement, Erlang's tables can be classified among the very best of their kind; having acquired a large circulation, they have been reprinted several times.

Femcifrede Logaritmer og Antilogaritmer — Five-Figure Tables of Logarithms and Anti-Logarithms.
Copenhagen, 1930.

Besides five-figure logarithms and antilogarithms, these tables also contain a seven-figure table of log $(1 + i)^n$. A part of the tables was in the press when Erlang died in 1929, and the edition was completed by Mr. *R. E. H. Rasmussen*, Ph. D. There are both Danish and English explanatory texts.

Erlang further published some shorter articles on the subject of mathematical tables:

Logaritmetabel og Regnestok. (*Tables of Logarithms and Slide Rule*).
Fysisk Tidsskrift, vol. 10, 1911—12, p. 285.

Prof. *Jul. Hartmann* had in an article about the logarithmic slide rule, published in the same journal on p. 230, praised the slide rule as being much better in respect of speed and serviceability than the four-figure table of logarithms, and recommended that slide rules be brought into use in school-teaching.

In his rejoinder, Erlang defends the use of the four-figure table of logarithms; he states, on the basis of some comparative speed trials he had made, that the slide rule is not much quicker (50 per cent. at most) than the four-figure table of logarithms which affords at least a ten times greater exactness.

Prof. Hartmann replies on p. 286 and admits to some extent the justness of Erlang's criticism.

Review of "Karl Pearson: Tables of the Incomplete Gamma-Function".
Skandinavisk Aktuarietidsskrift, vol. 6, 1923, p. 128.

In continuation of a mention of the methods of calculation employed by Pearson in the computation of his tables, Erlang mentions various tables of related functions and makes some remarks upon different interpolation formulae and methods of numerical integration.

Om et Par nye Multiplikationstabeller; en udvidet Anmeldelse. (*On Some New Multiplication Tables. An Extended Review*).
Matematisk Tidsskrift A, vol. 38, 1927, p. 115.

Erlang reviews herein a couple of small multiplication tables and mentions in this connexion various old and new tables of similar kind. He also states some reflections on calculation technique and nomography, and finally gives a copious list of literature on calculation technique.

Erlang has further written reviews of some compound interest tables in Nyt Tidsskrift for Matematik A, vol. 26, 1915, p. 38, and vol. 29, 1918, p. 40.

C. Other Mathematical Works.

Erlang was keenly interested in, and had a great knowledge of, many other branches of the science of mathematics than those mentioned under Sections A and B. The below-cited works, however, can give the reader but a faint idea of this fact, as Erlang, apart from answering the University Mathematical Prize Question for 1902—03, has published only 3 small articles on different subjects, all dating from his early years before his employment at the Copenhagen Telephone Company.

The annual prize questions of the University of Copenhagen are intended for being attempted by young men of science under 30 years of age, and the answers received are not published. Fully satisfactory answers are rewarded with gold medals, while such answers as are not considered worthy of gold medals but nevertheless bear witness of good scientific qualifications may be rewarded with a minor distinction called "Accessit".

Translated into English, the wording of the mathematical prize question for the year 1902—03 was as follows:

"Find in Huygens's works and recently published epistles his solutions of such problems as are now solved by means of the differential and integral calculus, and explain these solutions."

Erlang's work as a teacher did not leave him time enough to give a complete treatment of the prize question. He expresses his regret for this fact in his answer which he modestly calls an "attempt at an answer".

The judging committee, consisting of the professors *T. N. Thiele*, *H. G. Zeuthen* and *Julius Petersen*, states in its judgment[1]) that Erlang's work cannot be recognized as an adequate answer; that it, however, on account of its excellent and complete treatment of Huygens's mathematically most important work "Horologium oscillatorium" should be rewarded with "Accessit".

Lidt om det grafiske Korrespondensprincip. (*Something about the Principle of Graphical Correspondence*).
Nyt Tidsskrift for Matematik B, vol. 17, 1906, p. 58.

In a treatise on graphic curves[2]), Prof. *C. Juel* had set forth, and proved, the following theorem: "If there exists a continuous dependence between points X and points Y on a straight line, so that there are p different points Y corresponding to each point X, and q different points X corresponding to each point Y, and if the two directions of orientation are opposite, there will be $p + q$ common points".

Erlang gives in his article two new proofs of this theorem and mentions some applications of it.

Om Definitionen af Cirkelperiferiens Længde. (*On the Definition of the Length of the Perimeter of a Circle*).
Nyt Tidsskrift for Matematik A, vol. 18, 1907, p. 40.

In this short article Erlang makes some remarks upon the definition of the length of the circumference of a circle by means of inscribed and circumscribed polygons.

Flerfoldsvalg efter rene Partilister. (*Manifold Polling based upon Pure Party Lists*).
Nyt Tidsskrift for Matematik B, vol. 18, 1907, p. 82.

For the apportionment of the p available candidates' seats according to the respective numbers of votes a_1, a_2, a_3, ... secured by the different parties involved in an election, Erlang enumerates the following condi-

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1) Indbydelsesskrift til Kjøbenhavns Universitets Aarsfest i Anledning af Hans Majestæt Kongens Fødselsdag den 8. April 1904, p. 67.

2) *C. Juel:* Indledning i Læren om de grafiske Kurver. Det Kgl. Danske Videnskabernes Selskabs Skrifter, 6. Række, naturvidensk. og mathemat. Afd. 10, Bd. 1, Copenhagen, 1899, p. 1.

tions as being necessary and sufficient to ensure the election against any unfair results:

1. It must be impossible for any party to gain anything by some of its adherents' failing to vote.
2. It must be impossible for any party to gain anything by some of its adherents' giving their vote to the list of another party.

3a. When two minor parties coalesce into one major party, it must be impossible for either of the original parties to lose anything in consequence hereof (when the major party disunites into two, it must be impossible for either of the minor parties to gain anything).

3b. When two parties coalesce into one, it must be impossible to gain more than one seat in consequence hereof (when one party disunites into two, it must be impossible to lose more than one seat).

Erlang proves that these conditions can be satisfied only by using the method of calculation suggested by *V. d'Hondt*[1]); by this method, the seats are apportioned according to the magnitudes of the incomplete quotients obtained by dividing a_1, a_2, a_3, ... by a divisor d chosen such that the sum of the quotients becomes p.

Appendix 1. Erlang's Interconnexion Formula.

In his treatise: "The Application of the Theory of Probabilities in Telephone Administration" (p. 172) Erlang published — in Section 3c and Table 3 — his noteworthy interconnexion formula without proving it.

A deduction of this formula, using the principle of statistic equilibrium, will be given in the following.

Let it be assumed that altogether x trunks are provided, and that the selectors employed have k contacts, $k < x$, so that any one call is given access only to k of the x trunks; furthermore we presuppose that the holding times are distributed exponentially. Erlang now assumes an arrangement consisting in that the total traffic y offered to the x trunks is divided into so many equally large groups as to make one group for each of the ways in which it is possible to select k out of the x trunks and hunt over these k trunks. As k trunks can be chosen in $\binom{x}{k}$ ways from among the x, and the k trunks can be hunted over in $k!$ different manners of following, the traffic y must be divided into altogether $\binom{x}{k} \cdot k! = \frac{x!}{(x-k)!}$ equal groups.

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1) *V. d'Hondt:* Exposé du système pratique de représentation proportionelle adopté par le comité de l'association réformiste belgique. Gent, 1885.

Divided in this manner the traffic may be regarded as being distributed absolutely at random over the x trunks.

Let us now consider a call originating at an instant when altogether r out of the x trunks are busy. The call has access to k particular trunks among the x trunks, and we want to find the probability that the call will be lost, which will happen only if all k trunks are busy.

If $r < k$, the k trunks cannot all be busy, and accordingly the probability is zero.

If $r \geqq k$, the sought probability will, owing to the fact that we — as already mentioned — may consider the r busy trunks distributed absolutely at random over the x trunks, be identical with the ratio of the number of ways in which k trunks can be chosen out of r trunks to the number of ways in which k trunks can be chosen out of x trunks. Accordingly, the sought probability is $\frac{\binom{r}{k}}{\binom{x}{k}}$.

In order to determine the probabilities $S_0, S_1, S_2, \ldots, S_x$ that 0, 1, 2, ..., x of the x trunks will be busy, we consider an infinitely short time interval dt and assume that r trunks will be busy at the end of this interval, the corresponding probability being S_r. Now, this assumption can be satisfied in 3 ways only, *viz.*: —

1) if $r - 1$ trunks were busy at the beginning of dt, probability: S_{r-1}, and a call resulting in the occupation of a new trunk has originated during the time dt; the probability of this is, according to the above, $y \cdot dt$ for $r - 1 < k$, and $y \cdot \left(1 - \frac{\binom{r-1}{k}}{\binom{x}{k}}\right) \cdot dt$ for $r - 1 \geqq k$; or

2) if $r + 1$ trunks were busy at the beginning of dt, probability: S_{r+1}, and 1 of these $r + 1$ calls has terminated during the time dt, probability: $(r + 1)\, dt$; or

3) if r trunks were busy at the beginning of dt, probability: S_r, and no call resulting in a new occupation has originated and none of the r calls in progress has terminated during the time dt, probability: $1 - y \cdot dt - r \cdot dt$ for $r < k$, and $1 - y \cdot \left(1 - \frac{\binom{r}{k}}{\binom{x}{k}}\right) \cdot dt - r \cdot dt$ for $r \geqq k$.

Thus, using the principle of statistic equilibrium, we have:

$$S_r = S_{r-1} \cdot y \cdot dt + S_{r+1} \cdot (r+1) \cdot dt + S_r \cdot (1 - y \cdot dt - r \cdot dt) \qquad \text{for } r < k,$$

$$S_r = S_{r-1} \cdot y \cdot \left(1 - \frac{\binom{r-1}{k}}{\binom{x}{k}}\right) \cdot dt + S_{r+1} \cdot (r+1) \cdot dt$$

$$+ S_r \cdot \left(1 - y \cdot \left(1 - \frac{\binom{r}{k}}{\binom{x}{k}}\right) \cdot dt - r \cdot dt\right) \qquad \text{for } r \geqq k,$$

or:

$$(r+1) \cdot S_{r+1} = (y + r) \cdot S_r - y \cdot S_{r-1} \qquad \text{for } r < k,$$

$$(r+1) \cdot S_{r+1} = \left(y \cdot \left(1 - \frac{\binom{r}{k}}{\binom{x}{k}}\right) + r\right) \cdot S_r - y \cdot \left(1 - \frac{\binom{r-1}{k}}{\binom{x}{k}}\right) \cdot S_{r-1} \text{ for } r \geqq k.$$

Putting $r + 1 = 1, 2, 3, \ldots, x$, we obtain the following system of equations:

$$S_1 = y \cdot S_0$$

$$2\, S_2 = (y + 1) \cdot S_1 - y \cdot S_0$$

. . . .

$$k \cdot S_k = (y + k - 1) \cdot S_{k-1} - y \cdot S_{k-2}$$

$$(k+1) \cdot S_{k+1} = \left(y \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) + k\right) \cdot S_k - y \cdot S_{k-1}$$

$$(k+2) \cdot S_{k+2} = \left(y \cdot \left(1 - \frac{\binom{k+1}{k}}{\binom{x}{k}}\right) + k + 1\right) \cdot S_{k+1} - y \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot S_k$$

. . . .

$$x \cdot S_x = \left(y \cdot \left(1 - \frac{\binom{x-1}{k}}{\binom{x}{k}}\right) + x - 1\right) \cdot S_{x-1} - y \cdot \left(1 - \frac{\binom{x-2}{k}}{\binom{x}{k}}\right) \cdot S_{x-2}.$$

By successive addition of these equations, we get

$$S_1 = y \cdot S_0$$

$$2\,S_2 = y \cdot S_1$$

$$\cdots\cdots\cdots\cdot$$

$$k \cdot S_k = y \cdot S_{k-1}$$

$$(k+1) \cdot S_{k+1} = y \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot S_k$$

$$(k+2) \cdot S_{k+2} = y \cdot \left(1 - \frac{\binom{k+1}{k}}{\binom{x}{k}}\right) \cdot S_{k+1}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$x \cdot S_x = y \cdot \left(1 - \frac{\binom{x-1}{k}}{\binom{x}{k}}\right) \cdot S_{x-1},$$

which leads to:

$$S_1 = y \cdot S_0$$

$$S_2 = \frac{y^2}{2!} \cdot S_0$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$S_k = \frac{y^k}{k!} \cdot S_0$$

$$S_{k+1} = \frac{y^{k+1}}{(k+1)!} \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot S_0 = \frac{y^{k+1}}{(k+1)!} \cdot N_{k+1} \cdot S_0$$

$$S_{k+2} = \frac{y^{k+2}}{(k+2)!} \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot \left(1 - \frac{\binom{k+1}{k}}{\binom{x}{k}}\right) \cdot S_0 = \frac{y^{k+2}}{(k+2)!} \cdot N_{k+2} \cdot S_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_x = \frac{y^x}{x!} \cdot \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot \left(1 - \frac{\binom{k+1}{k}}{\binom{x}{k}}\right) \cdots \left(1 - \frac{\binom{x-1}{k}}{\binom{x}{k}}\right) \cdot S_0 = \frac{y^x}{x!} \cdot N_x \cdot S_0,$$

where we have introduced the notations:

$$N_{k+r} = \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) \cdot \left(1 - \frac{\binom{k+1}{k}}{\binom{x}{k}}\right) \cdots \left(1 - \frac{\binom{k+r-1}{k}}{\binom{x}{k}}\right), \quad r = 1, 2, \ldots, x-k.$$

From the relation

$$S_0 + S_1 + S_2 + \cdots + S_x = 1$$

we finally obtain:

$$S_0 = \frac{1}{1 + y + \frac{y^2}{2!} + \cdots + \frac{y^k}{k!} + \frac{y^{k+1}}{(k+1)!} \cdot N_{k+1} + \cdots + \frac{y^x}{x!} \cdot N_x},$$

by which the sought probabilities $S_0, S_1, S_2, \ldots, S_x$ are determined.

The probability of loss can now be determined directly, since we have found in the foregoing that the probability that a call will be lost when altogether r out of the x trunks are busy is zero for $r < k$, and $\frac{\binom{r}{k}}{\binom{x}{k}}$ for $r \geqq k$. The probability of loss is, accordingly,

$$B = S_k \cdot \frac{\binom{k}{k}}{\binom{x}{k}} + S_{k+1} \cdot \frac{\binom{k+1}{k}}{\binom{x}{k}} + \cdots + S_x \cdot \frac{\binom{x}{k}}{\binom{x}{k}}$$

$$= S_0 \cdot \left\{ \frac{y^k}{k!} \cdot \frac{\binom{k}{k}}{\binom{x}{k}} + \frac{y^{k+1}}{(k+1)!} \cdot N_{k+1} \cdot \frac{\binom{k+1}{k}}{\binom{x}{k}} + \cdots + \frac{y_x}{x!} \cdot N_x \cdot \frac{\binom{x}{k}}{\binom{x}{k}} \right\}.$$

Now we furthermore introduce the notations:

$$T_k = 1 - N_{k+1} = 1 - \left(1 - \frac{\binom{k}{k}}{\binom{x}{k}}\right) = \frac{\binom{k}{k}}{\binom{x}{k}},$$

$$T_{k+r} = N_{k+r} - N_{k+r+1} = N_{k+r} - N_{k+r} \cdot \left(1 - \frac{\binom{k+r}{k}}{\binom{x}{k}}\right) = N_{k+r} \cdot \frac{\binom{k+r}{k}}{\binom{x}{k}},$$

$$T_x = N_x, \qquad r = 1, 2, \ldots, x - k - 1,$$

thus obtaining

$$B = S_0 \cdot \left\{ \frac{y^k}{k!} \cdot T_k + \frac{y^{k+1}}{(k+1)!} \cdot T_{k+1} + \cdots + \frac{y^x}{x!} \cdot T_x \right\},$$

whence, finally, by introducing the value of S_0 found above,

$$B = \frac{\frac{y^k}{k!} \cdot T_k + \frac{y^{k+1}}{(k+1)!} \cdot T_{k+1} + \cdots + \frac{y^x}{x!} \cdot T_x}{1 + y + \frac{y^2}{2!} + \cdots + \frac{y^k}{k!} + \frac{y^{k+1}}{(k+1)!} \cdot N_{k+1} + \cdots + \frac{y^x}{x!} \cdot N_x}.$$

This formula for the probability of loss is identical with the formula given by Erlang, which only differs in form from this expression. In order to give the above expression the same appearance as Erlang's formula, we note that, in consequence of the relation

$$\frac{\binom{k+s}{k}}{\binom{x}{k}} = \frac{\binom{x+s}{x}}{\binom{x+s}{x-k}},$$

the expression for N_{k+r} may also be written:

$$N_{k+r} = \left(1 - \frac{\binom{x}{x}}{\binom{x}{x-k}}\right) \cdot \left(1 - \frac{\binom{x+1}{x}}{\binom{x+1}{x-k}}\right) \cdots \left(1 - \frac{\binom{x+r-1}{x}}{\binom{x+r-1}{x-k}}\right), \quad r = 1, 2, \ldots, x-k,$$

which is the form given by Erlang.

Further, we put

$$N_0 = N_1 = N_2 = \cdots = N_k = 1,$$
$$T_0 = T_1 = T_2 = \cdots = T_{k-1} = 0.$$

Finally, we multiply the expression for B by e^{-y} in numerator and denominator and put $P_s = e^{-y} \cdot \frac{y^s}{s!}$.

In this manner we obtain

$$B = \frac{T_0 \cdot P_0 + T_1 \cdot P_1 + \cdots + T_x \cdot P_x}{N_0 \cdot P_0 + N_1 \cdot P_1 + \cdots + N_x \cdot P_x},$$

which is the formula given by Erlang.

This formula is greatly interesting in that it gives the exact solution of an interconnecting-problem for arbitrary numbers of trunks, x, and contacts, k, whereas no such generally valid formulae for the loss in the commonly used grading-systems are known.

Erlang terms the interconnecting-method, on which his formula is based, the ideal method; and he considers this method likely to give minimum loss. On the whole, Erlang may possibly be right in conjecturing this; nevertheless it is not correct in all cases, as it will appear from a comparison with other gradings in cases where the number of trunks x is so small that it is possible to carry out the calculations. Thus, when Erlang's formula for the values $x = 3$, $k = 2$ is compared with a 2-contact-grading with 2 groups, consisting of 2 individual circuits and 1 common circuit, where the traffic is $\frac{1}{2}\,y$ in either of the 2 groups, we find the following values of the loss B for different values of y:

$y =$	0.1	0.5	1.0	2.0	3.0	5.0
Erlang's Formula.....	0.00161	0.03390	0.10638	0.26415	0.39130	0.55746
2-Group-Grading	0.00130	0.03153	0.10494	0.26705	0.39600	0.56199

In this case Erlang's method evidently gives a slightly greater loss than the 2-group-grading for values of y less than about 1.3.

Erlang gives two approximative formulae, one of which is applicable for very small values of y; for $y \ll 1$ it is sufficient to include the term containing the lowest power of y in numerator and denominator, i.e. $\frac{y^k}{k!} \cdot T_k$ and 1, respectively, thus obtaining

$$B \sim \frac{y^k}{k!} \cdot T_k = \frac{y^k}{k!} \cdot (1 - N_{k+1}) = \frac{y^k}{k!} \cdot \frac{1}{\binom{x}{k}} = y^k \cdot \frac{(x-k)\,!}{x\,!}.$$

The other approximative formula is applicable in cases where x and y are so great in comparison with k that the probability that a call will find each of the k contacts busy, with sufficient approximation can be regarded as constant and equal to the average traffic per circuit, α, so that

$$B \sim \alpha^k.$$

The average traffic per circuit is $\alpha = \frac{y \cdot (1 - B)}{x}$; for small values of B, obviously $\alpha \sim \frac{y}{x}$, and therefore:

$$B \sim \left(\frac{y}{x}\right)^k.$$

The last mentioned approximation formula forms the basis of *G. F. O'Dell's* well-known method of calculating the loss in ordinary gradings.[1])

[1]) *G. F. O'Dell:* Outline of the Trunking Aspect of Automatic Telephony, Journal of the Institution of Electrical Engineers, vol. 65, 1927, p. 185.

Appendix 2. Approximative Formulae for Loss and "Improvement".

In his dissertation "On the Rational Determination of the Number of Circuits" (p. 216), Erlang has — without proofs — given two approximative formulae (Formulae (3), p. 217, and (7), p. 220) which express the "improvement" in lost-traffic systems and waiting-time systems, respectively, by means of the Gaussian Normal Function.

A deduction of these formulae and related formulae for the loss B and the average waiting time M is given in the following.

In the case of a group of x circuits with an intensity of traffic y, the loss as determined by Erlang's B-formula is

$$B(x) = \frac{\frac{y^x}{x!}}{1 + y + \frac{y^2}{2!} + \cdots + \frac{y^x}{x!}}, \tag{1}$$

or, multiplying in numerator and denominator by e^{-y},

$$B(x) = \frac{P(x)}{P(0) + P(1) + \cdots + P(x)}, \tag{1a}$$

where $P(x)$ is Poisson's function:

$$P(x) = \frac{y^x}{x!} \cdot e^{-y}. \tag{2}$$

As we shall show in the following, this Poisson function can be expressed, with an approximation that improves as y increases, by means of the Gaussian Normal Function

$$\phi(h) = \frac{1}{\sqrt{2\,\pi}} \cdot e^{-\frac{h^2}{2}}. \tag{3}$$

The variable in this function is $h = \frac{\delta}{\sigma}$, where δ is the deviation from the mean value and σ is the standard deviation.

It is well known that the Poisson function (2) has the mean value y and the standard deviation $\sqrt{y}$, that is to say,

$$\delta = x - y.$$

and

$$\sigma = \sqrt{y}, \quad \text{or} \quad y = \sigma^2. \tag{3a}$$

The variable, h, of the Normal function must therefore be

$$h = \frac{\delta}{\sigma} = \frac{x - y}{\sqrt{y}},$$

whence

$$x = y + h \cdot \sqrt{y} = \sigma^2 + h \cdot \sigma. \tag{3b}$$

Introducing the expressions (3a) and (3b) into the Poisson function (2) we obtain

$$P(x) = \frac{y^x}{x!} \cdot e^{-y} = \frac{\sigma^{2 \cdot (\sigma^2 + h\sigma)}}{(\sigma^2 + h\sigma)!} \cdot e^{-\sigma^2}.$$

Now we replace the factorial $(\sigma^2 + h\,\sigma)!$ by its Stirling approximation, as it is well known that

$$m! = \sqrt{2\,\pi} \cdot m^{m+\frac{1}{2}} \cdot e^{-m} \cdot \left(1 + \frac{1}{12m} + \cdots\right).$$

Thus we get

$$P(x) = \frac{\sigma^{2 \cdot (\sigma^2 + h\sigma)} \cdot e^{-\sigma^2}}{\sqrt{2\,\pi} \cdot (\sigma^2 + h\sigma)^{\sigma^2 + h\sigma + \frac{1}{2}} \cdot e^{-\sigma^2 - h\sigma} \cdot \left(1 + \frac{1}{12 \cdot (\sigma^2 + h\,\sigma)} + \cdots\right)}$$

$$= \frac{e^{h\sigma}}{\sqrt{2\,\pi} \cdot \sigma \cdot \left(1 + \frac{h}{\sigma}\right)^{\sigma^2 + h\sigma + \frac{1}{2}}} \cdot \left(1 - \frac{1}{12\,\sigma^2} + \cdots\right).$$

In order to obtain an expansion of $P(x)$ in a series arranged according to powers of $\frac{1}{\sigma}$, we take the natural logarithm of the above expression:

$$\log P(x) = h\sigma - \log(\sqrt{2\,\pi} \cdot \sigma) - (\sigma^2 + h\sigma + \tfrac{1}{2}) \cdot \log\left(1 + \frac{h}{\sigma}\right) + \log\left(1 - \frac{1}{12\,\sigma^2} + \cdots\right),$$

whence, for $\sigma > h$, we get the expansion

$$\log P(x) = h\sigma - \log(\sqrt{2\pi} \cdot \sigma) - (\sigma^2 + h\sigma + \tfrac{1}{2}) \cdot \left(\frac{h}{\sigma} - \frac{h^2}{2\sigma^2} + \frac{h^3}{3\sigma^3} - \frac{h^4}{4\sigma^4} + \cdots\right) - \frac{1}{12\,\sigma^2} + \cdots$$

$$= -\log(\sqrt{2\,\pi} \cdot \sigma) - \frac{h^2}{2} - \frac{1}{6\,\sigma}(3\,h - h^3) - \frac{1}{12\,\sigma^2} \cdot (1 - 3\,h^2 + h^4) + \cdots,$$

and accordingly

$$P(x) = \frac{1}{\sqrt{2\,\pi}\cdot\sigma}\cdot e^{-\frac{h^2}{2}}\cdot e^{-\frac{1}{6\,\sigma}(3h-h^3)-\frac{1}{12\sigma^2}\cdot(1-3h^2+h^4)+\cdots}$$

$$= \frac{1}{\sqrt{2\,\pi}}\cdot e^{-\frac{h^2}{2}}\cdot\left(\frac{1}{\sigma}-\frac{1}{6\,\sigma^2}(3h-h^3)-\frac{1}{72\sigma^3}(6-27h^2+12h^4-h^6)+\cdots\right).$$

Introducing the expression (3) we finally find the approximative formula:

$$P(x) \sim \phi(h)\cdot\left(\frac{1}{\sigma}-\frac{1}{6\,\sigma^2}(3\,h-h^3)-\frac{1}{72\,\sigma^3}(6-27\,h^2+12\,h^4-h^6)\right). \quad (4)$$

The sum $P(0)+P(1)+\cdots+P(x)$ contained in the denominator of the expression (1a) for $B(x)$ can now be determined by means of Euler's summation formula:

$$P(0)+P(1)+\cdots+P(x) =$$

$$\int_0^x P(x)\cdot dx+\tfrac{1}{2}(P(0)+P(x))+\tfrac{1}{12}(P'(x)-P'(0))-\tfrac{1}{720}\cdot(P'''(x)-P'''(0))+\cdots,$$

in which we replace $P(x)$ with the approximation (4), which we will call $F(h)$. For the lower limit of integration we obtain from (3b) the value $h = -\sigma$, corresponding to $x = 0$. For large values of σ ($\sigma > 5$), this quantity can be replaced by $h = -\infty$, and the terms $P(0)$, $P'(0)$, ... will furthermore become negligible. We find, moreover, from (3b) that $dx = \sigma\cdot dh$, so that we get

$$P(0)+P(1)+\cdots+P(x) \sim$$

$$\int_{-\infty}^{h} F(h)\cdot\sigma\cdot dh+\tfrac{1}{2}\cdot F(h)+\frac{1}{12\,\sigma}\cdot F'(h)-\frac{1}{720\,\sigma^3}\cdot F'''(h)+\cdots.$$

Putting $\phi(h) = \phi$ and $\int_{-\infty}^{h}\phi(h)\,dh = \phi_{-1}$ we find:

$$\int_{-\infty}^{h} F(h)\cdot\sigma\cdot dh \sim \phi_{-1}+\phi\cdot\left(\frac{1}{6\,\sigma}\cdot(1-h^2)-\frac{1}{72\,\sigma^2}\cdot(6\,h-7\,h^3+h^5)\right),$$

$$\tfrac{1}{2}F(h) \sim \phi\cdot\left(\frac{1}{2\,\sigma}-\frac{1}{12\,\sigma^2}\cdot(3\,h-h^3)\right),$$

$$\frac{1}{12\,\sigma}\cdot F'(h) \sim -\phi\cdot\frac{h}{12\,\sigma^2},$$

and thus

$$P(0)+P(1)+\cdots+P(x)\sim\phi_{-1}+\phi\left(\frac{1}{6\,\sigma}\cdot(4-h^2)-\frac{1}{72\sigma^2}\cdot(30h-13h^3+h^5)\right),$$

whence

$$\frac{1}{P(0)+P(1)+\cdots+P(x)}\sim\frac{1}{\phi_{-1}}\cdot\left(1-\frac{\phi}{\phi_{-1}}\cdot\left(\frac{1}{6\,\sigma}\cdot(4-h^2)\right.\right.$$

$$\left.\left.-\frac{1}{72\,\sigma^2}\cdot(30\,h-13\,h^3+h^5)\right)+\left(\frac{\phi}{\phi_{-1}}\right)^2\cdot\frac{1}{36\,\sigma^2}\cdot(16-8\,h^2+h^4)\right).$$

Inserting this expression and the expression (4) for $P(x)$ in (1a), we get the following approximative formula for $B(x)$:

$$B(x)\sim\frac{\phi}{\phi_{-1}}\cdot\left(\frac{1}{\sigma}-\frac{1}{6\,\sigma^2}\cdot(3\,h-h^3)-\frac{1}{72\,\sigma^3}\cdot(6-27\,h^2+12\,h^4-h^6)\right)$$

$$-\left(\frac{\phi}{\phi_{-1}}\right)^2\cdot\left(\frac{1}{6\,\sigma^2}\cdot(4-h^2)-\frac{1}{24\,\sigma^3}\cdot(18h-9h^3+h^5)\right)$$

$$+\left(\frac{\phi}{\phi_{-1}}\right)^3\cdot\frac{1}{36\,\sigma^3}\cdot(16-8\,h^2+h^4) \qquad (5)$$

This formula, giving $B(x)$ with good accuracy for great values of y (see further particulars p. 125), is especially useful when it is desired to calculate $B(x)$ for so great values of y that they fall outside the range of the tables[1]).

For very great values of y, formula (5) may be reduced to the simplified form:

$$B(x)\sim\frac{\phi}{\sigma\cdot\phi_{-1}}. \qquad (5a)$$

This formula, however, gives only a rather rough approximation for such values of y as are important in practice.

For the sake of simplicity we shall confine ourselves to using formula (5a) as basis for the following derivation of approximation formulae for F_1, M, and F_2, these mostly having theoretical interest.

In Erlang's formula for the "improvement" in loss systems:

$$F_1=y\cdot B(x-1)-y\cdot B(x), \qquad (6)$$

we develop the difference by means of Taylor's series:

$$y\cdot B(x+u)-y\cdot B(x)=y\cdot u\cdot B'(x)+y\cdot\frac{u^2}{2}\cdot B''(x)+\cdots,$$

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[1]) *Conny Palm:* Table of the Erlang Loss Formula, Stockholm, 1947, gives B for values of y less than 100.

replacing $B(x)$ in this with the approximation (5a), i. e.,

$$y \cdot B(x) = \sigma^2 \cdot B(x) \sim f(h) = \sigma \cdot \frac{\phi}{\phi_{-1}}.$$

We have from (3b) that to the difference $u = \Delta x = -1$ corresponds $\Delta h = -\frac{1}{\sigma}$, and that $dx = \sigma \cdot dh$, so that we get

$$F_1 \sim f\left(h - \frac{1}{\sigma}\right) - f(h) = -\frac{1}{\sigma} \cdot f'(h) + \frac{1}{2\sigma^2} \cdot f''(h) - \cdots,$$

whence

$$F_1 \sim \frac{\phi}{\phi_{-1}} \cdot h + \left(\frac{\phi}{\phi_{-1}}\right)^2, \tag{7}$$

which is the approximation formula for F_1 as given by Erlang.

Determined by Erlang's formula, the average waiting time for a group of x circuits with traffic intensity y is

$$M(x) = \frac{1}{x - y} \cdot \frac{B(x-1) \cdot B(x)}{B(x-1) - B(x)}. \tag{8}$$

Using the relation

$$B(x-1) = \frac{x}{y} \cdot \frac{B(x)}{1 - B(x)} \tag{9}$$

we obtain

$$M(x) = \frac{x}{x - y} \cdot \frac{B(x)}{x - y + yB(x)}. \tag{8a}$$

Introducing the values (3a) and (3b) for x and y, and (5a) for $B(x)$, into (8a) we get the following approximation formula for $M(x)$:

$$M(x) \sim \frac{\sigma + h}{\sigma^2 \cdot h} \cdot \frac{\phi}{\phi + h\,\phi_{-1}}. \tag{10}$$

Treating Erlang's formula for the "improvement" in waiting time systems,

$$F_2 = y \cdot M(x-1) - y \cdot M(x), \tag{11}$$

in the same manner as the formula (6) for F_1, we get

$$F_2 \sim -\frac{1}{\sigma} \cdot g'(h) + \cdots,$$

where, according to (10),

$$y \cdot M(x) \sim g(h) = \frac{\sigma + h}{h} \cdot \frac{\phi}{\phi + h\phi_{-1}},$$

and so we find

$$F_2 \sim \frac{\phi^2 \cdot (1 + h^2) + \phi_{-1} \cdot \phi \cdot (2\,h + h^3)}{(\phi + h\phi_{-1})^2 \cdot h^2}, \tag{12}$$

which is the approximation formula for F_2 as given by Erlang.

We shall finally illustrate by an example the value of the approximative formulae found above, and we choose $x = 120$, $y = 100$. From (3a) and (3b) we find $\sigma = 10$ and $h = 2$, and from tables of the Gaussian Normal Law, $\phi = 0.053991$ and $\phi_{-1} = 0.977250$.

The exact value of $B(x)$ is $B(x) = 0.005690$, while formula (5) gives $B(x) = 0.005689$, the difference being only one unit in the sixth place; formula (5a) gives $B(x) \sim 0.005525$, i. e. an error of 2.9 %.

The exact value of F_1 is, according to (6), $F_1 = 0.1177$, whereas (7) gives $F_1 \sim 0.1135$, error: 3.6 %. (8) gives the exact value of $M(x)$, $M(x) = 0.001660$, while (10) gives $M(x) \sim 0.001613$, error: 2.8 %. The exact value of F_2, given by (11), is $F_2 = 0.0525$, whereas (12) gives $F_2 \sim 0.0401$, error: 23.6 %.

As it appears from this example, the formulae (5a), (7), (10), and (12) give a rough approximation only, even for so great a value of y; the approximation formula for F_2, especially, differs widely from the correct value. More exact formulae can, of course, be obtained by using formula (5) as basis instead of (5a), but they will be complicated and unpractical.

When calculating M, F_1, and F_2, it is easier and better to start from B and compute the said quantities by means of the simple, exact formulae (8) or (8a), (6), and (11). The value of B can be found in the table mentioned in the foot-note p. 123 or computed by means of formula (5) for greater values of y.

The following small table, showing, for $\sigma = 10$, the error k for different values of h, will serve to give an impression of the error in $B(x)$ resulting from the use of formula (5):

$B(x)$ for $y = 100$, $\sigma = 10$.

x	h	Exact Value	Formula (5)	Error k
100	0	0·075 700	0·075 703	+ 0·000 003
105	0·5	0·048 261	0·048 266	+ 0·000 005
110	1·0	0·027 463	0·027 468	+ 0·000 005
115	1·5	0·013 575	0·013 577	+ 0·000 002
120	2·0	0·005 690	0·005 689	— 0·000 001
125	2·5	0·001 989	0·001 989	0·000 000
130	3·0	0·000 576	0·000 577	+ 0·000 001

For other great values of σ the error will be $f \sim k \cdot \left(\frac{10}{\sigma}\right)^4$, where k for different values of h has the values stated in the table above. Any value of B computed by formula (5) may, if wanted, be corrected by subtracting the corresponding error f as given by this expression.

A SURVEY OF A. K. ERLANG'S ELECTROTECHNICAL WORKS

By H. L. HALSTRØM.

At the time when Erlang began his work in the service of the Copenhagen Telephone Company one of the problems that had special interest for the Company was the use of cables with artificially increased self-induction in order to improve the transmission quality. There were two solutions of this problem to choose between, *viz.* the method of increasing the self-induction of a circuit at uniformly spaced points, as suggested by Prof. *M. Pupin,* and the method of increasing the self-induction continuously, as suggested by *C. E. Krarup,* M. Sc., Chief Engineer of the Danish State Post & Telegraph Administration, and *J. L. W. V. Jensen,* M. Sc., Ph. D., Engineer-in-Chief to the Copenhagen Telephone Company; consequently it became part of Erlang's duties to carry out a great deal of calculations of various kinds, respecting *e. g.* the profitableness of different systems, the optimum interval between Pupin coils, the maximum reduction in attenuation by Krarup's method, ideal loading, &c. Most of these works are purely calculative even though they comprise several theoretical works also. Only one of the latter has been published, *viz.*:

9. *An Elementary Treatise on the Main Points of the Theory of Telephone Cables*, p. 233[1]).

   Published in Danish under the title of
   Hovedpunkterne af Teorien for Telefonkabler i elementær Fremstilling.
   Elektroteknikeren, vol. 7, 1911, p. 139.

After a historical, mathematical, and physical introduction to this work, Erlang deduces in a simple manner the principal formulae concerning the infinitely long, homogeneous cable; the results are then applied especially to cables with artificial, continuously distributed self-induction.

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[1]) The numbers prefixed in this survey to the titles of Erlang's reprinted works correspond to the numbers of the reprints in the present book, to which also the suffixed page numbers have reference.

It is shown that any length of cable is characterized by 4 principal constants, only 3 of which need be known as all 4 principal constants are linearly interdependent. The relationship existing between the principal constants and some constants mentioned in the foregoing is demonstrated, and rules of calculation are given for the connecting of cables in series. Next, the coil-loaded, or Pupin, cables are mentioned, and the cooperation of the receiving instrument with the line, and finally the application of the theory to the measuring methods.

As an example of Erlang's works on other subjects may be mentioned the following:

10. *An Elementary Theoretical Study of the Induction Coil in a Subscriber's Telephone Apparatus*, p. 253.
    First published in Danish:
    Transformatoren i et Telefonapparat, en elementær teoretisk Undersøgelse.
    Elektroteknikeren, vol. 10, 1914, p. 169.
    Later published in French:
    Etude théorique élémentaire sur le transformateur d'un appareil téléphonique.
    La Vie Technique et industrielle, vol. 9, 1927, octobre.

Having formulated, in this paper, the necessary assumptions, Erlang sets forth the basic equations that give the conditions with which a serviceable induction coil must comply; the theory is then applied to an example.

Being occupied with the theoretical problems presented by telephone cables, Erlang soon found himself wanting an instrument for measuring the transmission constants of cables, and this led to the construction of his "Complex Compensator". This compensator, constituting a decided improvement as compared with complex compensators of earlier date, is described in:

11. *New Alternating-Current Compensation Apparatus for Telephonic Measurements*, p. 261.
    Journal of the Institution of Electrical Engineers, vol. 51, 1913, p. 794.
    First published in Danish:
    Et nyt Kompensationsapparat til Vekselstrømsmaalinger indenfor Telefonien.
    Elektroteknikeren, vol. 9, 1913, p. 157.

The apparatus contains two measuring wires, connected in parallel, which are supplied with current through resistors and inductors, respectively. Potentials can be tapped from the two measuring wires by means of sliding-contacts; the amplitude and phase of these potentials can be adjusted so as to compensate an unknown potential. The apparatus can be used for measuring impedances, for transmission measurements on telephone circuits, and for measuring frequencies.

Erlang's complex compensator is easy to set up and easy to operate; but the computation of the results takes some time, especially when measuring on a range of frequencies, as the voltage vectors refer to oblique coordinate axes whose angle varies with the frequency. By altering Erlang's compensator, Prof. *P. O. Pedersen* a few years later succeeded in constructing a compensator[1]) with voltage coordinates which remain rectangular for all frequencies. P. O. Pedersen's compensator, however, still has a disadvantage in that the scale by which abscissae and ordinates are measured varies with the frequency, though the device may be made direct-reading for a standard frequency; it has been put into practical form by Messrs. H. Tinsley & Co.

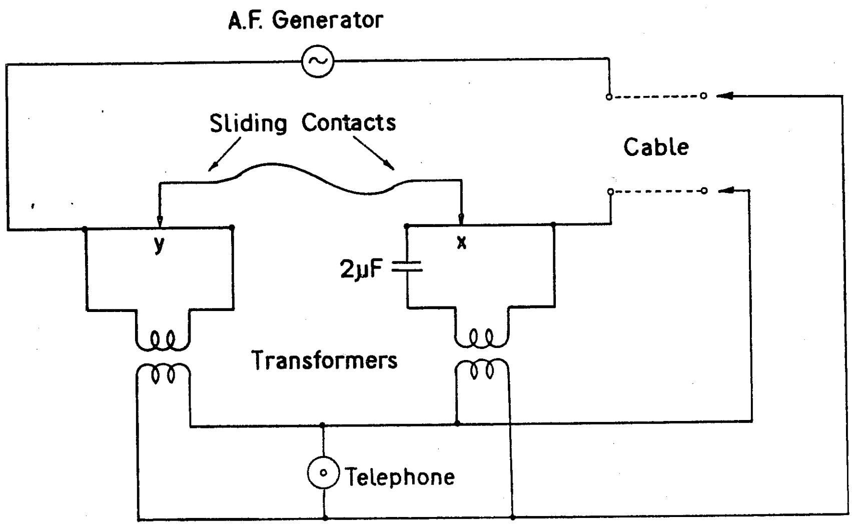
Not satisfied that his problem had been solved with the publication of the above-mentioned paper on the complex compensator, Erlang occasionally took it up for renewed treatment. Thus, among his unpublished works is the construction of a so-called "complex current-compensator", the principle of which will appear from the subjoined diagram showing the current-compensator when used for measuring the attenuation in a cable. A paper by *H. T. Stenby*[2]) contains a brief description of the apparatus.

The current-compensator contains two air-core transformers, the primaries of which are connected in series and get their supply of current through sliding-contacts as shown in the diagram. Owing to a capacitor being inserted in series with one primary, the secondaries can be tapped for currents, the amplitude and phase of which can be varied so as to compensate an unknown current. On the basis of the constants of the compensator it is possible, for any desired frequency, to work out a diagram arranged in such a manner that the attenuation can be read directly as a function of the positions of the sliding-contacts along the slide-wire.

[1]) *P. O. Pedersen:* A New Alternating-Current Potentiometer for Measurements on Telephone Circuits, Electrician, vol. 83, 1919, p. 523.

[2]) *H. T. Stenby:* Nogle Vekselstrømsmaalinger i Telefontekniken, Ingeniøren, vol. 43, 1934, II, p. 17.

This attenuation diagram consists of a family of Booth's lemniscates[1]), graphed relative to a system of oblique coordinate axes.



In continuation of his experiments with the current-compensator, Erlang had commenced the preliminary work in connexion with a new voltage-compensator of a modified construction, but he did not live to complete it.

At his death Erlang left several notes — especially on Pupin cables, but also on the balancing of two-wire repeater circuits and other problems — which were not intended for publication. The only electrotechnical studies published by Erlang are the three works commented upon above and reprinted in the present book; they will serve to illustrate Erlang's faculty of applying mathematical points of view to the solution of electrotechnical problems in telephony.

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[1]) These curves, the corresponding Cartesian equation of which is $(x^2 + y^2)^2 = a^2 x^2 \pm b^2 y^2$, are elliptic or hyperbolic according as the sign of the last member on the right side of the equation is + or —. In the latter case, the curves for $a^2 = b^2$ will be ordinary (Bernoulli's) lemniscates, $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$. The curves referred to above have been investigated by *J. Booth* in "A Treatise on some New Geometrical Methods", London, 1877, vol. I, p. 162 ff. The name of "Booth's lemniscates" was given by *G. Loria* (see: G. Loria: "Spezielle algebraische und transzendente ebene Kurven", 2. Aufl., 1910, vol. I, p. 134).

PRINCIPAL WORKS
OF
A. K. ERLANG

1. THE THEORY OF PROBABILITIES AND TELEPHONE CONVERSATIONS

First published in "Nyt Tidsskrift for Matematik" B, Vol. 20 (1909), p. 33.

Although several points within the field of Telephony give rise to problems, the solution of which belongs under the Theory of Probabilities, the latter has not been utilized much in this domain, so far as can be seen. In this respect the Telephone Company of Copenhagen constitutes an exception as its managing director, Mr. *F. Johannsen*, through several years has applied the methods of the theory of probabilities to the solution of various problems of practical importance; also, he has incited others to work on investigations of similar character. As it is my belief that some point or other from this work may be of interest, and as a special knowledge of telephonic problems is not at all necessary for the understanding thereof, I shall give an account of it below.

1. *The probability of a certain number of calls being originated during a certain interval of time.*

It is assumed that there is no greater probability of a call being attempted at one particular moment than at any other moment. Let a be the time interval given, n the average number of calls during the unit of time. We will find the probability S_0 of 0 calls being originated during the time a, and afterwards the probability S_x of exactly x calls being originated during the time a. As $\frac{na}{r}$ is the probability of calls during the time $\frac{a}{r}$ when r is infinitely great, $1-\frac{na}{r}$ is, on the same assumption, the probability of 0 attempts being made during the time $\frac{a}{r}$. Hence we have

$$S_0 = \lim_{r=\infty}\left(1-\frac{na}{r}\right)^r = e^{-na}. \qquad \text{(I)}$$

Now in order to find S_x, the time a can be divided into r equal elements where $r \geqq x$, and x of these elements chosen, which can be done in $C_{r,x}$

ways. We are now seeking, firstly, the probability of 1 call being originated during each of the x elements; secondly, the probability of no attempts being made during the remaining time, which is $\frac{a\,(r-x)}{r}$. The former probability is, for $r=\infty$, $\left(\frac{na}{r}\right)^x$; the latter is according to (I)

$$e^{-\frac{na\,(r-x)}{r}}.$$

Thus, we get $S_x = \lim\limits_{r=\infty} C_{r,x}\left(\frac{na}{r}\right)^x e^{-\frac{na\,(r-x)}{r}}$, or, as $\lim\limits_{r=\infty}\frac{C_{r,x}}{r^x}=\frac{1}{x!}$,

$$S_x = \frac{(na)^x}{x!}\,e^{-na}. \tag{II}$$

This formula becomes less complicated if we let m denote na, the average number of calls arriving during the time a. Then we have

$$S_x = \frac{m^x}{x!}\,e^{-m}. \tag{III}$$

2. *The Law of Distribution.*

When, in the formula thus found, x is allowed to assume the values of all whole numbers from 0 upwards, the formula will express a certain "law of error", or "law of distribution". It is at once obvious that the sum of all the probabilities is 1, as it should be; further, that the probabilities of an even number and of an odd number of calls are, respectively,

$$\frac{e^m + e^{-m}}{2\,e^m} \quad \text{and} \quad \frac{e^m - e^{-m}}{2\,e^m}.$$

The chief property of the law of distribution is that all "half-invariants" are equal to m (*T. N. Thiele:* Theory of Observations, London, 1903); here, I shall confine myself to showing that the mean square error is m. We get,

$$\left(\frac{1^2\,m}{1!}+\frac{2^2m^2}{2!}+\frac{3^2m^3}{3!}+\frac{4^2m^4}{4!}+\ldots\right)e^{-m}-m^2$$

$$=\left(\frac{m^2}{1!}+\frac{2\,m^3}{2!}+\frac{3\,m^4}{3!}\ldots + m + \frac{m^2}{1!}+\frac{m^3}{2!}+\frac{m^4}{3!}+\frac{m^5}{4!}+\ldots\right)e^{-m}-m^2$$

$$=(m^2e^m + me^m)\,e^{-m} - m^2 = m.$$

The simple suppositions leading to the simple formula (III) will not, of course, always be satisfied in practice. Let us suppose, *e. g.*, that a business firm has certain busy days every week corresponding to a mean value m_1, and certain less busy days corresponding to a mean value m_2. Let the busy portion of the week be p_1, and the less busy portion p_2, where $p_1 + p_2 = 1$. If it is desired here to express the variations in the number of calls from day to day in terms of one single law of distribution, we find that the mean value is

$$p_1 m_1 + p_2 m_2;$$

but the mean square error is

$$p_1 p_2 (m_1 - m_2)^2$$

greater than the mean value. As, however, the preceding simple theory proves, on the whole, to be corresponding fairly well with meter readings experienced, we shall stick to that in the following.

3. *Delay in answering of telephone calls.*

We will assume that each operator receives calls from a determinate group of subscribers only, the system being designed in such a way that she cannot get help from her neighbours even if she is occupied and they happen to be free at the moment. By choosing a suitable unit of time, we can fix the average at 1 call per unit of time. The establishing of a connexion lasts t units of time. If a call is originated while the operator is unoccupied, we shall here consider the delay in answering, or waiting time, as being non-existent (actually, a certain short space of time will pass before the signal is noticed). If, on the other hand, she is occupied with another call, then the calling subscriber will have to wait a certain time. The problem is now to determine the function $f(z)$, $f(z)$ representing the probability of the waiting time not exceeding z.

The probability that, at the moment a call arrives, the time having elapsed since the preceding call should be confined within the limits

$$y \text{ and } y + dy,$$

is $e^{-y} dy$. The probability that the waiting time of the preceding call has been less than $z + y - t$, is $f(z + y - t)$.

Hence, we obtain

$$f(z) = \int_{y=0}^{\infty} f(z + y - t)\, e^{-y}\, dy.$$

By differentiation with respect to z, this equation gives

$$f'(z) = \int_{y=0}^{\infty} f'(z + y - t)\, e^{-y}\, dy,$$

and by partial integration,

$$f(z) = f(z - t) + \int_{y=0}^{\infty} f'(z + y - t)\, e^{-y}\, dy.$$

Thus we have

$$f'(z) = f(z) - f(z - t). \tag{IV}$$

By integration, $f(z)$ can now be determined in a succession of intervals, on the assumption that $f(z) = 0$ for $z < 0$, jumps from 0 to $1 - t$ for $z = 0$, but varies continuously for all other values of z.

For

$$\begin{array}{l} 0 < z < t, \\ t < z < 2t, \\ 2t < z < 3t, \\ 3t < z < 4t, \\ \cdots\cdots \\ \cdots\cdots \\ nt < z < (n+1)\, t \end{array}$$

the results will then be, respectively,

$$f(z) = (1 - t)\, e^{z}$$

$$f(z) = (1 - t)\, (e^{z} + e^{z-t}\, (t - z))$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$f(z) = (1 - t) \left(e^{z} + \frac{e^{z-t}\, (t - z)}{1!} + \frac{e^{z-2t}\, (2t - z)^2}{2!} \cdots \frac{e^{z-nt}\, (nt - z)^n}{n!} \right)$$

This is easily proved by inserting in equation (IV) the value of $f(z)$ taken from the last of the above formulae, and the value of $f(z - t)$ taken from the last but one.

As to the numerical calculation, it will be advantageous to begin with making a table of the function

$$\frac{m^x}{x!}\, e^{-m}$$

for negative values of m (*e. g.* with intervals of 0.1), and for positive and integral values of x; for this purpose, one of the existing tables of log $x!$ will be a good help. The oldest of these — which is, also, still the best — is *C. F. Degen:* Tabularum enneas (Havniæ, 1824). The terms in the table of the function must then be summed along oblique lines, and the obtained sums multiplied by $1 - t$, to provide the values for the definitive table giving $f(z)$ as a function of z and t. A table of this kind is printed below (table 1).

Table 1. Values of $f(z)$.

z \ t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000	0.900	0.800	0.700	0.600	0.500	0.400	0.300	0.200	0.100	0.000
0.1	1.000	0.995	0.884	0.774	0.663	0.553	0.442	0.332	0.221	0.111	0.000
0.2	1.000	1.000	0.977	0.855	0.733	0.611	0.489	0.366	0.244	0.122	0.000
0.3	1.000	1.000	0.991	0.945	0.810	0.675	0.540	0.405	0.270	0.135	0.000
0.4	1.000	1.000	0.998	0.967	0.895	0.746	0.597	0.448	0.298	0.149	0.000
0.5	1.000	1.000	0.999	0.983	0.923	0.824	0.659	0.495	0.330	0.165	0.000
0.6	1.000	1.000	1.000	0.992	0.947	0.856	0.729	0.547	0.364	0.182	0.000
0.7	1.000	1.000	1.000	0.996	0.965	0.885	0.761	0.605	0.403	0.201	0.000
0.8	1.000	1.000	1.000	0.998	0.977	0.910	0.792	0.635	0.445	0.223	0.000
0.9	1.000	1.000	1.000	0.999	0.984	0.931	0.822	0.665	0.470	0.246	0.000
1.0	1.000	1.000	1.000	0.999	0.990	0.947	0.849	0.694	0.495	0.261	0.000
1.1	1.000	1.000	1.000	1.000	0.993	0.958	0.872	0.722	0.520	0.276	0.000
1.2	1.000	1.000	1.000	1.000	0.995	0.967	0.891	0.749	0.545	0.292	0.000
1.3	1.000	1.000	1.000	1.000	0.997	0.975	0.906	0.773	0.569	0.307	0.000
1.4	1.000	1.000	1.000	1.000	0.998	0.980	0.920	0.794	0.592	0.323	0.000
1.5	1.000	1.000	1.000	1.000	0.999	0.985	0.932	0.812	0.614	0.339	0.000
1.6	1.000	1.000	1.000	1.000	0.999	0.988	0.942	0.829	0.635	0.354	0.000
1.7	1.000	1.000	1.000	1.000	1.000	0.991	0.950	0.845	0.653	0.369	0.000
1.8	1.000	1.000	1.000	1.000	1.000	0.993	0.957	0.859	0.671	0.384	0.000
1.9	1.000	1.000	1.000	1.000	1.000	0.994	0.964	0.872	0.688	0.397	0.000
2.0	1.000	1.000	1.000	1.000	1.000	0.996	0.969	0.884	0.705	0.411	0.000

4. In order to facilitate the understanding of the preparation of the final table giving the values of $f(z)$, I shall give, in table 2, the values of the Poisson function

$$\frac{m^x}{x!} e^{-m}$$

for negative values of m. Incidentally, the values of $f(z)$ can be obtained in a different manner by means of a table giving the values of the said function for x being positive. In this case it is necessary, as previously mentioned, to add up all the terms placed along oblique lines, or diagonals; then, the number of terms to be considered is infinite, but most often convergence will be very rapid.

By multiplication by $1 - a$ is thus obtained directly, not the probability $f(z)$ of an inferior delay in answering, but the probability $1 - f(z)$ of a superior delay for a fixed value of z. This is easily proved by means of a theorem by *J. L. W. V. Jensen**), according to which $\frac{1}{1-a}$ is equal to the sums of the terms situated along oblique lines in a complete Poisson table, i. e. one comprising positive as well as negative values of m. As to denotations, I have here used a, the symbol employed by Mr. *Johannsen* who was the first to discuss theoretically the important question of delays in answering**).

For great values of z, an approximated (asymptotic) formula may be employed which simplifies the calculation:

$$S = 1 - f(z) = \frac{1-a}{a'-1} e^{-z\frac{a'-a}{a}},$$

where the figures a and a' ($a < 1 < a'$) are bound by the relation

$$a e^{-a} = a' e^{-a'}.$$

Finally, the average delay can be given the simple expression

$$M = \frac{a}{2(1-a)}.$$

It should be remembered that it is an essential presupposition for the results stated above that the calls be of constant duration.

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*) Acta mathematica, XXVI, 1902, p. 309.

**) The Post Office Electrical Engineers' Journal, October, 1910, p. 244, and January, 1911, p. 303.

Table 2.

Values of $\frac{m^x}{x!} e^{-m}$ for $m < 0$.

x \ m	—1.0	—0.9	—0.8	—0.7	—0.6	—0.5	—0.4	—0.3	—0.2	—0.1	—0.0	m / x
0	+2.718282	+2.459603	+2.225541	+2.013753	+1.822119	+1.648721	+1.491825	+1.349859	+1.221403	+1.105171	+1.000000	0
1	—2.718282	—2.213643	—1.780433	—1.409627	—1.093271	—0.824360	—0.596730	—0.404958	—0.244281	—0.110517		1
2	+1.359141	+0.996139	+0.712173	+0.493369	+0.327981	+0.206090	+0.119346	+0.060744	+0.024428	+0.005526		2
3	—0.453047	—0.298842	—0.189913	—0.115119	—0.065596	—0.034348	—0.015913	—0.006074	—0.001629	—0.000184		3
4	+0.113262	+0.067239	+0.037983	+0.020146	+0.009839	+0.004294	+0.001591	+0.000456	+0.000081	+0.000005		4
5	—0.022652	—0.012103	—0.006077	—0.002820	—0.001181	—0.000429	—0.000127	—0.000027	—0.000003			5
6	+0.003775	+0.001815	+0.000810	+0.000329	+0.000118	+0.000036	+0.000008	+0.000001				6
7	—0.000539	—0.000233	—0.000093	—0.000033	—0.000010	—0.000003						7
8	+0.000067	+0.000026	+0.000009	+0.000003	+0.000001							8
9	—0.000007	—0.000003	—0.000001									9
10	+0.000001											10

x \ m	—2.0	—1.9	—1.8	—1.7	—1.6	—1.5	—1.4	—1.3	—1.2	—1.1	—1.0	m / x
0	+7.389056	+6.685894	+6.049647	+5.473947	+4.953032	+4.481689	+4.055200	+3.669297	+3.320117	+3.004166	+2.718282	0
1	—14.778112	—12.703199	—10.889365	—9.305711	—7.924852	—6.722534	—5.677280	—4.770086	—3.984140	—3.304583	—2.718282	1
2	+14.778112	+12.068039	+9.800429	+7.909854	+6.339881	+5.041900	+3.974096	+3.100556	+2.390484	+1.817520	+1.359141	2
3	—9.852075	—7.643092	—5.880257	—4.482251	—3.381270	—2.520950	—1.854578	—1.343574	—0.956194	—0.666424	—0.453047	3
4	+4.926037	+3.630468	+2.646116	+1.904957	+1.352508	+0.945356	+0.649102	+0.436662	+0.286858	+0.183267	+0.113262	4
5	—1.970415	—1.379578	—0.952602	—0.647685	—0.432803	—0.283607	—0.181749	—0.113532	—0.068846	—0.040319	—0.022652	5
6	+0.656805	+0.436866	+0.285781	+0.183511	+0.115414	+0.070902	+0.042408	+0.024599	+0.013769	+0.007392	+0.003775	6
7	—0.187659	—0.118578	—0.073486	—0.044567	—0.026380	—0.015193	—0.008482	—0.004568	—0.002360	—0.001162	—0.000539	7
8	+0.046915	+0.028162	+0.016534	+0.009470	+0.005276	+0.002849	+0.001484	+0.000742	+0.000354	+0.000160	+0.000067	8
9	—0.010425	—0.005945	—0.003307	—0.001789	—0.000938	—0.000475	—0.000231	—0.000107	—0.000047	—0.000020	—0.000007	9
10	+0.002085	+0.001130	+0.000595	+0.000304	+0.000150	+0.000071	+0.000032	+0.000014	+0.000006	+0.000002	+0.000001	10
11	—0.000379	—0.000195	—0.000097	—0.000047	—0.000022	—0.000010	—0.000004	—0.000002	—0.000001			11
12	+0.000063	+0.000031	+0.000015	+0.000007	+0.000003	+0.000001						12
13	—0.000010	—0.000005	—0.000002	—0.000001								13
14	+0.000001	+0.000001										14

2. SOLUTION OF SOME PROBLEMS IN THE THEORY OF PROBABILITIES OF SIGNIFICANCE IN AUTOMATIC TELEPHONE EXCHANGES

First published in "Elektroteknikeren" Vol. 13 (1917) p. 5.

Summary. — Sections 1—7. First main problem: Systems without waiting arrangements. (Two different presuppositions.) Accompanied by Tables 1, 2, 3. Sections 8—9. Second main problem: Systems with waiting arrangement. (Two different presuppositions.) Accompanied by Tables 4, 5, 6, 7. Sections 10—12. Approximative methods, references, conclusion. Accompanied by Table 8.

1. First Main Problem. — Let us suppose that an automatic system is arranged in such a manner that there are provided x lines to take a certain number of subscribers. These x lines are said to be co-operative, or to constitute a "group" (or "team"). It is presupposed that all the lines disengaged are accessible. At present we will only speak of systems without waiting arrangements, *i. e.* systems in which the subscriber, when he finds that all x lines are engaged, replaces the receiver, and does not try to get connection again immediately. The probability of thus finding the lines engaged is called the loss, or degree of hindrance, and is here designated by B. With respect to the length of the conversations (sometimes called the holding-time), we will (for the present) suppose that it is constant, and it will be convenient to consider this quantity equal to 1 ("the natural time-unit"). With respect to the subscribers' calls, it is assumed that they are distributed quite accidentally throughout the time in question (*e. g.* that part of the day when the heaviest traffic usually occurs). This presupposition does not only imply that there must not be points of time within the period of time in consideration at which it may be expected in advance that there will be exceptionally many or few calls, but also that the calls must be mutually independent. In practice these presuppositions will, with great approximation, be fulfilled. The average number of calls per time-unit (intensity of traffic) is called y. The ratio of y to x, *i. e.* the traffic intensity per line, is designed by α; it is often called the efficiency of the group. We have to determine B (as a function of y and x). The exact expression for this is as follows:

$$B = \frac{\dfrac{y^x}{x!}}{1 + \dfrac{y}{1!} + \dfrac{y^2}{2!} + \ldots + \dfrac{y^x}{x!}} \tag{1}$$

as proved in the following sections (2—5).

2. The following proof may be characterised as belonging to the mathematical statistics, and is founded on the theory of "statistical equilibrium" — a conception which is of great value in solving certain classes of problems in the theory of probabilities. Let us consider a very great number of simultaneously operating groups of lines of the previously described kind (number of lines x, traffic intensity y). If we examine a separate group at a definite moment, we may describe its momentary condition by stating, firstly, how many of the x lines $(0, 1, 2, \ldots . x)$ are engaged; and secondly, how much there is left of each of the conversations in question. If we examine the same group a short time dt later, we will find that certain changes of two different kinds have taken place. On the one hand, the conversations which were nearly finished will now be over, and the others have become a little older. On the other hand, new calls may have been made, which, however, will have significance only if not all the lines are engaged. (The probability of a new call during the short time dt is ydt.) We assume that we examine in this manner not only one group, but a very great number of groups, both with respect to the momentary condition and the manner in which this alters. The state, of which we thus can get an accurate description, if we use a sufficiently large material, has the characteristic property that, notwithstanding the aforesaid individual alterations, it maintains itself, and, when once begun, remains unaltered, since the alterations of the different kinds balance each other. This property is called "statistic equilibrium".

3. Temporarily as a postulate, we will now set forth the following description of the state of statistical equilibrium.

The probabilities that $0, 1, 2, 3, \ldots . x$ lines are engaged are respectively—

$$\left.\begin{aligned}
S_0 &= \frac{1}{1+\frac{y}{1!}+\frac{y^2}{2!}+\ldots+\frac{y^x}{x!}} \\
S_1 &= \frac{\frac{y}{1}}{1+\frac{y}{1!}+\frac{y^2}{2!}+\ldots+\frac{y^x}{x!}} \\
S_2 &= \frac{\frac{y^2}{2}}{1+\frac{y}{1!}+\frac{y^2}{2!}+\ldots+\frac{y^x}{x!}} \\
&\ldots\ldots\ldots\ldots \\
S_x &= \frac{\frac{y^x}{x!}}{1+\frac{y}{1!}+\frac{y^2}{2!}+\ldots+\frac{y^x}{x!}}
\end{aligned}\right\} \quad (2)$$

where the sum of all the probabilities is 1, as it should be. And we further postulate for each of the $x+1$ aforesaid special conditions, that the still remaining parts of the current conversations ("remainders") will vary quite accidentally between the limits 0 and 1, so that no special value or combination of values is more probable than the others.

4. We shall prove that the thus described general state is in statistical equilibrium. For that purpose we must keep account of the fluctuations (increase and decrease), during the time dt, for the $x+1$ different states, beginning with the first two. The transition from the first state S_0 to the second state S_1 amounts to

$$S_0\, ydt,$$

while the transition from the second S_1 to the first S_0 amounts to

$$S_1 \cdot dt.$$

These quantities are according to (3) equal and thus cancel each other. Furthermore, the amount of transition from S_1 to S_2 is:

$$S_1 \cdot ydt,$$

and, conversely, the transition from S_2 to S_1 is:

$$S_2 \cdot 2 \cdot dt,$$

which two quantities also are equal and cancel each other.

Finally, we have

$$S_{x-1} \cdot y \cdot dt$$

and

$$S_x \cdot x \cdot dt,$$

which also cancel each other. The result is that the reciprocal changes which take place between the $x + 1$ different states during the time dt, compensate each other, so that the distribution remains unaltered. We still have to prove that neither will there be any alterations in the distribution of the magnitude of the remainders, *i. e.* that the decrease and increase, also in this respect, compensate each other.

5. Let us consider the cases in which the number of current conversations is n, and among these cases, more especially those in which the magnitudes of the n remainders lie, respectively, between the following limits:

$$t_1 \text{ and } t_1 + \varDelta_1,$$
$$t_2 \text{ and } t_2 + \varDelta_2,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$t_n \text{ and } t_n + \varDelta_n.$$

The probability of this is (according to Section 3):

$$\varDelta_1 \cdot \varDelta_2 \cdot \varDelta_3 \ldots \varDelta_n \cdot S_n.$$

During the time dt there may occur, in four different ways, both increase and decrease.

Firstly, transition to S_{n+1}; namely, if a call arrives; the probability of this will be:

$$\varDelta_1 \cdot \varDelta_2 \cdot \varDelta_3 \ldots \varDelta_n \cdot S_n \cdot y \cdot dt.$$

Secondly, transition from S_{n+1}; namely, if one among the $n + 1$ current conversations finishes during the time dt, and, thereafter, the n remainders lie between the above settled limits. The corresponding probability is:

$$\varDelta_1 \cdot \varDelta_2 \cdot \varDelta_3 \ldots \varDelta_n (n + 1) S_{n+1} \cdot dt,$$

which is equal to the preceding one.

Thirdly, transition from S_n itself; namely, if, among the n remainders, the $n - 1$ lie between the settled limits, and the one lies just below the lower limit in question, at a distance shorter than dt. The probability for this will be:

$$\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \ldots \Delta_n \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \ldots + \frac{1}{\Delta_n}\right) S_n \cdot dt.$$

Fourthly, transition to S_n itself; namely, if, among the n remainders, the $n - 1$ lie between the settled limits, and the one lies just below the upper limit, at a distance shorter than dt. The probability of this eventuality is obviously equal to the preceding one.

Thus, there is a balance. So it is proved by this that there will be statistical equilibrium. On the other hand, any other supposition than the one set forth in Section 3 will at once be seen to be inconsistent with statistic equilibrium. The formulæ in Section 3 are now proved, and thereby the proposition in Section 1 is also proved.

6. The above presupposition, that all conversations are of equal length, applies with great approximation to trunk-line conversations, but not, of course, to the usual local conversations. Now, a statistic investigation, which I have undertaken, shows that the duration of these conversations is ruled by a simple law of distribution, which may be expressed as follows:

The probability that the duration will exceed a certain time n is equal to

$$e^{-n},$$

when the average duration is taken to be equal to 1, as before. Or, in other words, the probability that a conversation which has been proceeding for some time is nearly finished, is quite independent of the length of the time which has already elapsed. The average number of conversations finished during the time dt (per current conversation) will be equal to dt. It is now easy to see that we must arrive at the same expression (1) for B as under the former presupposition, only that the proof becomes somewhat simpler, because it is necessary to take into account only the number of current conversations without paying any attention to their age. (It will appear from the following that the two aforesaid presuppositions do not lead to the same result in *all* problems.)

7. In Table 1 are shown some numerical values of the "loss" B as dependent of x and y (or α), and as given by the proposed theory.

In Table 2 the results of formula (1) are presented in another form, which is probably the one that is most useful in practice; x and B are here entry numbers, and the table gives y as a function of x and B.

In Table 3a only the first and second lines treat of systems with "pure" groups (to which formula (1) applies). The values given in the third line

Table 1.

Values of the Loss, or Grade of Service, B. (Formula (1), Section 1).

x	a	y	B
1	0.1	0.1	0.091
1	0.2	0.2	0.167
2	0.1	0.2	0.016
2	0.2	0.4	0.054
2	0.3	0.6	0.101
3	0.1	0.3	0.003
3	0.2	0.6	0.020
3	0.3	0.9	0.050
3	0.4	1.2	0.090
4	0.1	0.4	0.001
4	0.2	0.8	0.008
4	0.3	1.2	0.026
4	0.4	1.6	0.056
5	0.2	1.0	0.003
5	0.3	1.5	0.014
5	0.4	2.0	0.037
5	0.5	2.5	0.070
6	0.2	1.2	0.001
6	0.3	1.8	0.008
6	0.4	2.4	0.024
6	0.5	3.0	0.052
8	0.3	2.4	0.002
8	0.4	3.2	0.011
8	0.5	4.0	0.030
10	0.3	3	0.001
10	0.4	4	0.005
10	0.5	5	0.018
10	0.6	6	0.043
10	0.7	7	0.079
20	0.4	8	0.000
20	0.5	10	0.002
20	0.6	12	0.010
20	0.7	14	0.030
30	0.5	15	0.000
30	0.6	18	0.003
30	0.7	21	0.014
40	0.5	20	0.000
40	0.6	24	0.001
40	0.7	28	0.007

Table 2.

Values of the intensity of traffic, y, as a function of the number of lines, x, for a loss of 1, 2, 3, 4 ‰.

x	1 ‰	2 ‰	3 ‰	4 ‰
1	0.001	0.002	0.003	0.004
2	0.046	0.065	0.081	0.094
3	0.19	0.25	0.29	0.32
4	0.44	0.53	0.60	0.66
5	0.76	0.90	0.99	1.07
6	1.15	1.33	1.45	1.54
7	1.58	1.80	1.95	2.06
8	2.05	2.31	2.48	2.62
9	2.56	2.85	3.05	3.21
10	3.09	3.43	3.65	3.82
11	3.65	4.02	4.26	4.45
12	4.23	4.64	4.90	5.11
13	4.83	5.27	5.56	5.78
14	5.45	5.92	6.23	6.47
15	6.08	6.58	6.91	7.17
16	6.72	7.26	7.61	7.88
17	7.38	7.95	8.32	8.60
18	8.05	8.64	9.03	9.33
19	8.72	9.35	9.76	10.07
20	9.41	10.07	10.50	10.82
25	12.97	13.76	14.28	14.67
30	16.68	17.61	18.20	18.66
35	20.52	21.56	22.23	22.75
40	24.44	25.6	26.3	26.9
45	28.45	29.7	30.5	31.1
50	32.5	33.9	34.8	35.4
55	36.6	38.1	39.0	39.8
60	40.8	42.3	43.4	44.1
65	45.0	46.6	47.7	48.5
70	49.2	51.0	52.1	53.0
75	53.5	55.3	56.5	57.4
80	57.8	59.7	61.0	61.9
85	62.1	64.1	65.4	66.4
90	66.5	68.6	69.9	70.9
95	70.8	73.0	74.4	75.4
100	75.2	77.5	78.9	80.0
105	79.6	82.0	83.4	84.6
110	84.1	86.4	88.0	89.2
115	88.5	91.0	92.5	93.7
120	93.0	95.5	97.1	98.4

Table 3 a.

The "Loss" (in $^0/_{00}$) by 3 different arrangements (one with "Grading and Interconnecting").

y	3	4	5	6	7	8	9	10	11	12
1) $x = 10$, with 10 contacts	0.8	5.3	18.4	43.1	—	—	—	—	—	—
1) $x = 18$, with 18 contacts	—	—	—	—	0.2	0.9	2.9	7.1	14.8	26.5
3). $x = 18$, with 10 contacts	—	—	—	—	1.1	3.1	7.4	15.1	26.8	42.8

Table 3 b.

Values of α and y by different arrangements for a loss of 1 $^0/_{00}$.

	α	y
$x = 10$; 10 contacts	0.31	3.1
$x = 18$; 10 -	0.38	6.9
$x = \infty$; 10 -	0.50	—

correspond to a different system, in which a special arrangement, the so-called "grading and interconnecting", is used. We may describe this arrangement as follows:

The number of contacts of the selectors (here ten) is less than the number of lines (here eighteen) in the "group". Thus each call searches not all eighteen but only ten lines. It is hereby presupposed (for the sake of simplicity) that the ten lines are each time accidentally chosen, out of the eighteen, and that they are tested one after the other according to an arbitrary selection. The method of calculation here to be used may be considered as a natural extension of the method which leads to formula (1), but it is, of course, a little more complicated. A few results of this kind of calculating are given in the two Tables 3 a and 3 b. Finally, I want to point out that the systems for "grading and interconnecting" being used in practice at present, which I, however, do not know in detail, are said to deviate a little from the description given here, and, therefore, it may be expected that they will give somewhat less favourable results.

8. Second Main Problem. — The problem to be considered now concerns systems with waiting arrangements. Here, the problem to be solved is determining the probability $S\ (> n)$ of a waiting time greater than an arbitrary number n, greater than or equal to zero. The last case is the one which is most frequently asked for. In the same manner we define $S(<n)$ where $S(<n)+S(>n)=1$. Furthermore, we may ask about the average waiting time M. We shall answer these questions in the following. Here, too, we may begin by assuming that the duration of the conversations is constant and equal to 1. The accurate treatment of this case gives rise to rather difficult calculations, which, however, are unavoidable. Among other things, we find that we cannot use the same formula for $S(>n)$ for all values of n, but we must distinguish between the various successive "periods", or spaces of time of the length 1. In practice, however, the first period will, as a rule, be the most important. I shall content myself by giving, without proof, the necessary formulæ for the cases of $x=1$, 2, and 3, and then (chiefly for the purpose of showing the possibility of carrying out the practical calculation) the corresponding numerical results, also for $x=1$, 2, 3. Formulæ and results for $x=1$ have already been published in an article in "Nyt Tidsskrift for Mathematik", B, 20, 1909. The formulæ for greater values of x, *e. g.* $x=10$, $x=20$ are quite analogous to those given here.

COLLECTION OF FORMULAE

Presupposition: the duration of conversations is constant and equal to 1

Denotations:

x is the number of co-operating lines

y is the intensity of traffic (average number of calls during unit of time)

$\frac{y}{x}=a$

$S(>n)$ is the probability of a waiting time greater than n

$S(<n)$ is the probability of a waiting time less than, or equal to n

$ny=z$

$z-y=u$

$z-2y=v$, et cetera.

$M=$ the average waiting time.

I. Formulae for the case of $x=1$:

a) First period, $0<n<1$:

$$S(<n)=a_0\cdot e^z,$$

where $a_0=1-a$

b) Second period, $1<n<2$:

$$S(<n)=(b_1-b_0u)\,e^u$$

where $\begin{cases} b_1=a_0e^y \\ b_0=a_0 \end{cases}$

c) Third period, $2 < n < 3$:

$$S(<n) = (c_2 - c_1 v + \tfrac{1}{2} c_0 v^2)\, e^v$$

where $\begin{cases} c_2 = (b_1 - b_0 y)\, e^y \\ c_1 = b_1 \\ c_0 = b_0 \end{cases}$

et cetera.

$$M = \frac{1}{y}\left((1 - b_1) + (1 - c_2) + (1 - d_3) + \ldots\right) = \frac{1}{2} \cdot \frac{a}{1 - a}$$

II. Formulae for the case of $x = 2$:

a) First period, $0 < n < 1$:

$$S(<n) = (a_1 - a_0 z)\, e^z$$

where $\begin{cases} a_1 = 2\,(1 - a)\, \dfrac{a}{a - \beta} \\ a_0 = -2\,(1 - a)\, \dfrac{\beta}{a - \beta} \end{cases}$,

β denoting the negative root of the equation

$$\beta e^{-\beta} = -\, a e^{-a}.$$

b) Second period, $1 < n < 2$:

$$S(<n) = (b_3 - b_2 u + \tfrac{1}{2} b_1 u^2 - \tfrac{1}{6} b_0 u^3)\, e^u$$

where $\begin{cases} b_3 = (a_1 - a_0 y)\, e^y \\ b_2 = a_0 e^y \\ b_1 = a_1 \\ b_0 = a_0 \end{cases}$

c) Third period, $2 < n < 3$:

$$S(<n) = (c_5 - c_4 v + \tfrac{1}{2} c_3 v^2 - \tfrac{1}{6} c_2 v^3 + \tfrac{1}{24} c_1 v^4 - \tfrac{1}{120} c_0 v^5)\, e^v$$

where $\begin{cases} c_5 = (b_3 - b_2 y + \tfrac{1}{2} b_1 y^2 - \tfrac{1}{6} b_0 y^3)\, e^y \\ c_4 = (b_2 - b_1 y + \tfrac{1}{2} b_0 y^2)\, e^y \\ c_3 = b_3 \\ c_2 = b_2 \\ c_1 = b_1 \\ c_0 = b_0 \end{cases}$

et cetera.

$$M = \frac{1}{y}\left((1 - b_2) + (1 - b_3) + (1 - c_4) + (1 - c_5) + (1 - d_6) + (1 - d_7) + \ldots\right)$$

III. Formulae for the case of $x = 3$

a) First period, $0 < n < 1$:

$$S\,(< n) = (a_2 - a_1 z + \tfrac{1}{2}\, a_0 z^2)\, e^z$$

where
$$\begin{cases} a_2 = 3\,(1-\alpha)\,\dfrac{\alpha^2}{(\alpha-\beta)\,(\alpha-\gamma)} \\[2ex] a_1 = -\,3\,(1-\alpha)\,\dfrac{\alpha\,(\beta+\gamma)}{(\alpha-\beta)\,(\alpha-\gamma)} \\[2ex] a_0 = 3\,(1-\alpha)\,\dfrac{\beta\cdot\gamma}{(\alpha-\beta)\,(\alpha-\gamma)}, \end{cases}$$

as
$$\beta\cdot e^{-\beta} = \alpha\cdot e^{-\alpha}\cdot k$$
$$\gamma\cdot e^{-\gamma} = \alpha\cdot e^{-\alpha}\cdot k^2$$

We understand by k a complex value of $\sqrt[3]{1}$.

b) Second period, $1 < n < 2$:

$$S\,(< n) = (b_5 - b_4 u + \tfrac{1}{2}\, b_3 u^2 - \tfrac{1}{6}\, b_2 u^3 + \tfrac{1}{24}\, b_1 u^4 - \tfrac{1}{120}\, b_0 u^5)\, e^u,$$

where
$$\begin{cases} b_5 = (a_2 - a_1 y + \tfrac{1}{2}\, a_0 y^2)\, e^y \\ b_4 = (a_1 - a_0 y)\, e^y \\ b_3 = a_0 e^y \\ b_2 = a_2 \\ b_1 = a_1 \\ b_0 = a_0 \end{cases}$$

c) Third period, $2 < n < 3$:

$$S\,(< n) =$$
$$(c_8 - c_7 v + \tfrac{1}{2} c_6 v^2 - \tfrac{1}{6} c_5 v^3 + \tfrac{1}{24} c_4 v^4 - \tfrac{1}{120} c_3 v^5 + \tfrac{1}{720} c_2 v^6 - \tfrac{1}{5040} c_1 v^7 + \tfrac{1}{40320} c_0 v^8)\, e^v,$$

where
$$\begin{cases} c_8 = (b_5 - b_4 y + \tfrac{1}{2}\, b_3 y^2 - \tfrac{1}{6}\, b_2 y^3 + \tfrac{1}{24}\, b_1 y^4 - \tfrac{1}{120}\, b_0 y^5)\, e^y \\ c_7 = (b_4 - b_3 y + \tfrac{1}{2}\, b_2 y^2 - \tfrac{1}{6}\, b_1 y^3 + \tfrac{1}{24}\, b_0 y^4)\, e^y \\ c_6 = (b_3 - b_2 y + \tfrac{1}{2}\, b_1 y^2 - \tfrac{1}{6}\, b_0 y^3)\, e^y \\ c_5 = b_5 \\ c_4 = b_4 \\ c_3 = b_3 \\ c_2 = b_2 \\ c_1 = b_1 \\ c_0 = b_0, \end{cases}$$

et cetera.

$$M = \frac{1}{y}\,((1-b_3) + (1-b_4) + (1-b_5) + (1-c_6) + (1-c_7) + (1-c_8) + \ldots\,).$$

Table 4. $(x = 1)$.

a \ n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.950	0.955	0.960	0.964	0.969	0.974	0.979	0.984	0.989	0.994	0.999
0.10	0.900	0.909	0.918	0.927	0.937	0.946	0.956	0.965	0.975	0.985	0.995
0.15	0.850	0.863	0.876	0.889	0.903	0.916	0.930	0.944	0.958	0.973	0.988
0.20	0.800	0.816	0.833	0.849	0.867	0.884	0.902	0.920	0.939	0.958	0.977
0.25	0.750	0.769	0.788	0.808	0.829	0.850	0.871	0.893	0.916	0.939	0.963
0.30	0.700	0.721	0.743	0.766	0.789	0.813	0.838	0.864	0.890	0.917	0.945
0.35	0.650	0.673	0.697	0.722	0.748	0.774	0.802	0.830	0.860	0.891	0.922
0.40	0.600	0.624	0.650	0.677	0.704	0.733	0.763	0.794	0.826	0.860	0.895
0.45	0.550	0.575	0.602	0.630	0.658	0.689	0.720	0.754	0.788	0.825	0.863
0.50	0.500	0.526	0.553	0.581	0.611	0.642	0.675	0.710	0.746	0.784	0.824
0.55	0.450	0.475	0.502	0.531	0.561	0.592	0.626	0.661	0.699	0.738	0.780
0.60	0.400	0.425	0.451	0.479	0.508	0.540	0.573	0.609	0.646	0.686	0.729
0.65	0.350	0.374	0.399	0.425	0.454	0.484	0.517	0.552	0.589	0.628	0.670
0.70	0.300	0.322	0.345	0.370	0.397	0.426	0.457	0.490	0.525	0.563	0.604
0.75	0.250	0.269	0.290	0.313	0.337	0.364	0.392	0.423	0.456	0.491	0.529
0.80	0.200	0.217	0.235	0.254	0.275	0.298	0.323	0.350	0.379	0.411	0.445
0.85	0.150	0.163	0.178	0.194	0.211	0.229	0.250	0.272	0.296	0.322	0.351
0.90	0.100	0.109	0.120	0.131	0.143	0.157	0.172	0.188	0.205	0.225	0.246
0.95	0.050	0.055	0.060	0.066	0.073	0.080	0.088	0.097	0.107	0.118	0.129
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 5. $(x = 2)$.

a \ n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.995	0.996	0.997	0.998	0.998	0.999	0.999	1.000	1.000	1.000	1.000
0.10	0.982	0.985	0.988	0.991	0.993	0.995	0.997	0.998	0.999	1.000	1.000
0.15	0.962	0.968	0.974	0.980	0.985	0.989	0.993	0.996	0.998	0.999	1.000
0.20	0.936	0.946	0.956	0.965	0.973	0.980	0.987	0.992	0.995	0.998	0.999
0.25	0.903	0.918	0.933	0.946	0.958	0.969	0.978	0.986	0.992	0.996	0.998
0.30	0.866	0.886	0.905	0.922	0.939	0.953	0.967	0.978	0.986	0.992	0.995
0.35	0.825	0.849	0.872	0.895	0.915	0.935	0.952	0.966	0.978	0.987	0.991
0.40	0.779	0.808	0.835	0.862	0.888	0.911	0.933	0.952	0.967	0.978	0.985
0.45	0.730	0.762	0.794	0.825	0.855	0.883	0.909	0.932	0.951	0.966	0.975
0.50	0.677	0.712	0.748	0.783	0.817	0.849	0.880	0.907	0.931	0.949	0.961
0.55	0.621	0.658	0.697	0.735	0.773	0.809	0.844	0.875	0.903	0.925	0.941
0.60	0.561	0.601	0.641	0.681	0.722	0.762	0.800	0.836	0.868	0.894	0.913
0.65	0.499	0.539	0.580	0.622	0.664	0.706	0.748	0.787	0.822	0.852	0.875
0.70	0.435	0.473	0.514	0.556	0.599	0.642	0.685	0.726	0.764	0.798	0.824
0.75	0.368	0.404	0.442	0.483	0.525	0.568	0.611	0.653	0.692	0.728	0.757
0.80	0.298	0.331	0.366	0.403	0.442	0.482	0.523	0.564	0.603	0.639	0.669
0.85	0.227	0.254	0.283	0.315	0.348	0.384	0.420	0.457	0.493	0.527	0.556
0.90	0.153	0.173	0.195	0.219	0.244	0.272	0.300	0.330	0.359	0.387	0.412
0.95	0.077	0.088	0.101	0.114	0.129	0.144	0.161	0.179	0.196	0.214	0.230
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 6. ($x = 3$).

a \ n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.10	0.996	0.997	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000
0.15	0.989	0.992	0.994	0.996	0.997	0.998	0.999	1.000	1.000	1.000	1.000
0.20	0.976	0.982	0.987	0.991	0.994	0.996	0.998	0.999	1.000	1.000	1.000
0.25	0.958	0.967	0.975	0.983	0.988	0.993	0.996	0.998	0.999	1.000	1.000
0.30	0.933	0.948	0.960	0.971	0.980	0.987	0.992	0.996	0.998	0.999	0.999
0.35	0.903	0.923	0.940	0.956	0.969	0.979	0.987	0.993	0.996	0.998	0.999
0.40	0.866	0.892	0.915	0.936	0.953	0.968	0.980	0.988	0.993	0.996	0.998
0.45	0.823	0.855	0.884	0.910	0.934	0.953	0.969	0.980	0.988	0.993	0.995
0.50	0.775	0.812	0.847	0.879	0.908	0.933	0.953	0.969	0.980	0.987	0.991
0.55	0.720	0.762	0.803	0.841	0.876	0.906	0.932	0.952	0.967	0.977	0.983
0.60	0.660	0.706	0.752	0.795	0.835	0.872	0.903	0.929	0.948	0.962	0.971
0.65	0.595	0.644	0.693	0.740	0.786	0.827	0.864	0.895	0.920	0.938	0.951
0.70	0.524	0.574	0.625	0.676	0.725	0.771	0.813	0.849	0.879	0.902	0.919
0.75	0.448	0.497	0.548	0.600	0.651	0.700	0.746	0.787	0.821	0.849	0.871
0.80	0.367	0.413	0.461	0.511	0.562	0.611	0.659	0.702	0.740	0.773	0.799
0.85	0.282	0.322	0.364	0.409	0.455	0.501	0.547	0.590	0.629	0.663	0.693
0.90	0.192	0.222	0.255	0.291	0.328	0.366	0.405	0.442	0.477	0.509	0.538
0.95	0.098	0.115	0.134	0.155	0.177	0.201	0.225	0.249	0.273	0.295	0.316
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

9. There still remains the problem of investigating the magnitude of the waiting times in systems with waiting arrangement under the second presupposition, namely, that the durations of the conversations vary in the manner already described in Section 6.

Here we find, without difficulty, the following two formulæ:

$$S\,(>0) = c \tag{3}$$

$$S\,(>n) = c \cdot e^{-(x-y)\,n} \tag{4}$$

where

$$c = \frac{\dfrac{y^x}{x!} \cdot \dfrac{x}{x-y}}{1 + \dfrac{y}{1!} + \dfrac{y^2}{2!} + \ldots + \dfrac{y^{x-1}}{(x-1)!} + \dfrac{y^x}{x!} \cdot \dfrac{x}{x-y}} \tag{5}$$

while x and y have the same significance as before, and the average duration of a conversation is equal to 1. The formula is exact for all values of $n \geqq 0$.

Table 7.

Systems with Waiting Arrangement (Second Presupposition). Values of $S\,(>n)$ and M.

x	a	y	$S\,(>0)$	$S\,(>0.1)$	$S\,(>0.2)$	M
1	0.1	0.1	0.100	0.091	0.084	0.111
1	0.2	0.2	0.200	0.185	0.170	0.250
2	0.1	0.2	0.018	0.015	0.013	0.010
2	0.2	0.4	0.067	0.057	0.049	0.042
2	0.3	0.6	0.138	0.120	0.104	0.099
3	0.1	0.3	0.004	0.003	0.002	0.001
3	0.2	0.6	0.024	0.019	0.015	0.010
3	0.3	0.9	0.070	0.057	0.046	0.033
3	0.4	1.2	0.141	0.118	0.099	0.078
4	0.1	0.4	0.001	0.001	0.000	0.000
4	0.2	0.8	0.010	0.007	0.005	0.003
4	0.3	1.2	0.037	0.028	0.022	0.013
4	0.4	1.6	0.091	0.072	0.056	0.038
5	0.2	1.0	0.004	0.003	0.002	0.001
5	0.3	1.5	0.020	0.014	0.010	0.006
5	0.4	2.0	0.060	0.044	0.033	0.020
5	0.5	2.5	0.130	0.102	0.079	0.052
6	0.2	1.2	0.002	0.001	0.001	0.000
6	0.3	1.8	0.011	0.007	0.005	0.003
6	0.4	2.4	0.040	0.026	0.018	0.011
6	0.5	3.0	0.099	0.073	0.054	0.033
8	0.3	2.4	0.004	0.002	0.001	0.001
8	0.4	3.2	0.018	0.011	0.007	0.004
8	0.5	4.0	0.059	0.040	0.026	0.015
10	0.3	3	0.001	0.001	0.000	0.000
10	0.4	4	0.009	0.005	0.003	0.001
10	0.5	5	0.036	0.022	0.013	0.007
10	0.6	6	0.102	0.068	0.046	0.026
10	0.7	7	0.222	0.165	0.122	0.074
20	0.4	8	0.000	0.000	0.000	0.000
20	0.5	10	0.004	0.001	0.001	0.000
20	0.6	12	0.024	0.011	0.005	0.003
20	0.7	14	0.094	0.052	0.028	0.016
22	0.5	11.0	0.002	0.001	0.000	0.000
22	0.6	13.2	0.018	0.007	0.003	0.002
22	0.7	15.4	0.081	0.042	0.022	0.012
30	0.5	15	0.000	0.000	0.000	0.000
30	0.6	18	0.007	0.002	0.001	0.001
30	0.7	21	0.044	0.018	0.007	0.005
40	0.5	20	0.000	0.000	0.000	0.000
40	0.6	24	0.002	0.000	0.000	0.000
40	0.7	28	0.022	0.007	0.002	0.002

For the average waiting time we get the formula:

$$M = \frac{c}{x - y} \tag{6}$$

The numerical calculation causes no special difficulty. It ought, perhaps, to be pointed out that, both here and in Section 8, it is presupposed that the waiting calls are despatched in the order in which they have been received. If this does not take place in practice, it will, of course, have a slight effect upon the value of $S\ (> n)$, but not at all on the value of M, neither on $S\ (> 0)$.

10. Approximative Formulæ. — The *exact* formulæ given above are throughout so convenient, that there is hardly any need of approximative formulæ. This does not, however, apply to the formulæ which concern the second main problem, first presupposition. Therefore, it may be worth while to mention a couple of approximative methods which quickly lead to a serviceable result, at least in such cases as have the greatest practical significance.

One of these methods has already been used by me, at the request of Mr. *P. V. Christensen*, Assistant Chief Engineer to the Copenhagen Telephone Company, for calculating the explicit tables given in the first pages of his fundamental work, "The Number of Selectors in Automatic Telephone Systems" (published in the Post Office Electrical Engineers' Journal, October, 1914, p. 271; also in "Elektroteknikeren", 1913, p. 207; "E. T. Z.", 1913, p. 1314).

Since the method used has not been described in full, I shall here say a few words about the same. The probability of just x calls being originated during a period of time for which the average number is y, is, as well known, under the usual presuppositions (Section 1):

$$e^{-y} \frac{y^x}{x!}$$

The mathematical theorem here used is due to *S. D. Poisson* ("Recherches sur la probabilité, etc.", 1835), and has later been studied by *L. v. Bortkewitsch* ("Das Gesetz der kleinen Zahlen", 1898). The function has been tabulated by the latter (*loc. cit.*), and later by *H. E. Soper* ("Biometrica", vol. X, 1914; also in *K. Pearson* "Tables for Statisticians, etc.", 1914).

Thus the probability of x or more calls during the mentioned period of time is:

$$P = e^{-y} \frac{y^x}{x!} + e^{-y} \frac{y^{x+1}}{(x+1)!} + e^{-y} \frac{y^{x+2}}{(x+2)!} + \ldots \tag{7}$$

Table 8.

Values of y as a function of x, for $P = 0.001 — 0.002 — 0.003 — 0.004$.

x	1 ‰	2 ‰	3 ‰	4 ‰
1	0.001	0.002	0.003	0.004
2	0.045	0.065	0.08	0.09
3	0.19	0.24	0.28	0.31
4	0.42	0.52	0.58	0.63
5	0.73	0.86	0.95	1.02
6	1.11	1.27	1.38	1.46
7	1.52	1.72	1.85	1.95
8	1.97	2.20	2.35	2.47
9	2.45	2.72	2.89	3.02
10	2.96	3.25	3.45	3.60
11	3.49	3.82	4.03	4.19
12	4.04	4.41	4.62	4.81
13	4.61	5.00	5.24	5.43
14	5.19	5.61	5.87	6.07
15	5.79	6.23	6.51	6.72
16	6.40	6.86	7.16	7.38
17	7.03	7.51	7.82	8.06
18	7.66	8.17	8.49	8.74
19	8.31	8.84	9.18	9.44
20	8.96	9.51	9.87	10.14
21	9.61	10.20	10.57	10.84
22	10.28	10.89	11.27	11.56
23	10.96	11.59	11.98	12.28
24	11.65	12.29	12.70	13.01
25	12.34	13.00	13.42	13.74
30	15.87	16.6	17.1	17.4
35	19.5	20.4	20.9	21.3
40	23.5	24.2	24.8	25.2
45	27.1	28.1	28.7	29.2
50	30.9	32.0	32.7	33.2
55	34.9	36.0	36.8	37.3
60	38.9	40.1	40.9	41.4
65	43.0	44.2	45.0	45.6
70	47.0	48.3	49.2	49.8
75	51.0	52.4	53.3	54.0
80	55.1	56.6	57.6	58.3
85	59.3	60.9	61.8	62.5
90	63.5	65.1	66.1	66.9
95	67.7	69.3	70.4	71.1
100	71.9	73.6	74.7	75.5
105	76.2	77.9	79.0	79.8
110	80.4	82.2	83.3	84.2
115	84.7	86.6	87.7	88.5
120	89.0	90.9	92.1	93.0

It will then be seen that P, in many cases, *viz.* when y is not unproportionally great, will be a good approximate value for the fraction of the calls which will find all the lines engaged (or for "the probability of not getting through"). Thus P in the case of exchanges without waiting arrangements approximates the "*loss*", and here gives obviously a somewhat too great value. In exchanges with waiting arrangement P approximates the quantity $S\,(>0)$, the probability of delay, and gives here a somewhat too small value. Or, if it is the fraction named above which is given beforehand, as is generally the case in practice, where often the value 0.001 is used, the formula will show the connexion between y and x. The values of y found in this manner (see Table 8) will never deviate 5 per cent. from the correct values in systems without waiting arrangement; never 1 per cent. in systems with waiting arrangement (both presuppositions), if we take the named, frequently used value of $P = 0.001$. Possible intermediate systems between the two main classes of exchanges may, of course, be treated with good results according to the same method.

If, in systems with waiting arrangement, we ask about the number of waiting times beyond a certain limit n, $S\,(>n)$, an extension of the same formula may be used, y being replaced by $y\,(1-n)$. The method is best suited for small values of n, and the error goes to the same side as mentioned above. Furthermore, it may be mentioned in this connexion that if we use, in the case considered, the formulæ following from presupposition No. 2, instead of those based upon presupposition No. 1, the errors thus introduced will be small, as a rule; they go, this time, in such a direction that we get too *great* values for $S\,(>0)$ and $S\,(>n)$; or, if it is y which is sought, there will be too small values for y.

11. It will be too lengthy to describe or mention, in this connexion, all the systematic practical experiments and measurements (also only partly published), which of late years have been made, partly by the firms in question (especially, Siemens and Halske, and Western Electric Co.), partly by others, or such purely empirical formulæ as have thus been set forth. On the other hand, it would be incorrect to neglect one or two interesting theoretical works from recent years, which directly concern one of the problems treated above. In his doctor's thesis, Mr. *F. Spiecker* ("Die Abhängigkeit des erfolgreichen Fernsprechanrufes von der Anzahl der Verbindungsorgane", 1913), has indicated a method for determining the loss in systems without waiting arrangement, which (as he himself admits) is not quite free from errors, and which, besides, is so complicated that it can hardly find application in practice. It should be emphasized, however, that the results in the cases in which the author has completed his calculations, lie very near the results of formula (1) given above. In the same work is also given an approximative formula, which can best be compared with the formula for P (Section 10 above). The difference is exclusively due to a somewhat deviating, and probably less practical, formulation of the problem. Mr. *W. H. Grinsted,* in his treatise, "A Study of Telephone Traffic Problems, etc." (Post Office Electrical Engineers' Journal, April 1915), presents a solution of the same problem. Since this solution has, probably, by many readers as well as by the author himself, been considered mathematically exact, it should be noticed that an error has occurred in the derivation of the formula in question and that, for this reason, the formula gives rather incorrect results. It should be added that the treatise is a reprint of an older work from 1907 (which I have not had opportunity to examine). In spite of the faulty results, Grinsted's work is, however, when its time of publication is considered, of no little merit.

12. In closing this article, I feel called upon to render my best thanks to Mr. *F. Johannsen,* Managing Director of the Copenhagen Telephone Co., not only for his interest in the investigations recorded here, but also for his energetic initiative in starting rational and scientific treatment of many different problems in connexion with telephone traffic. I also owe many thanks to Mr. *J. L. W. V. Jensen,* Engineer-in-Chief to the same Company, for his valuable assistance especially in the treatment of some mathematical difficulties.

3. TELEPHONE WAITING TIMES

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1. Formulating the problem; how to reach the solution.

For some years, all the experts — particularly, perhaps, in Denmark — have been aware that the application of the theory of probabilities constitutes the only possible way of attaining fully rational methods in telephone administration. This holds good with respect to the exploitation of lines and the utilization of the work of operators, and it is especially valid for the newest, more or less automatic telephone systems. I have treated some of the problems of primary importance in this connexion in an article in "Elektroteknikeren", 1917 (and later in "Elektrotechnische Zeitschrift", 1918, and "The Post Office Electrical Engineers' Journal", 1918), in which, however, I have omitted — for the sake of brevity — some of the proofs, and stated only the resulting formulae and numerical expressions. I shall mention only one important problem here, *viz.* that of finding the probability that the delay in answering, or waiting time, shall not exceed a given quantity z, expressed as a function of z. The given quantities, then, are: — The number of available lines, x; the duration of the call, t; and the intensity of traffic, y (*i. e.* the average number of conversations proceeding simultaneously, or, in a different wording, the average number of calls during the time t. It is presupposed that $y < x$; also, that a calling subscriber who cannot be connected at once because all x lines are occupied, will always wait — possibly in a "queue" with other waiting subscribers — until he gets through. The duration of the calls t is here assumed to be constant; this assumption holds good with respect to trunk calls, but is less accurate in the case of local calls, the latter generally being of rather variable duration which gives rise to a problem of a kind somewhat different from the one we propose to deal with here. For convenience, the unit of time should either be considered equal to the duration of calls, or it should be chosen in such a way that there will be an average of 1 call per unit of time; the latter method is the one preferred here, and thus $t = y$.

The solution of our problem can be reached, or, at least, approached in rather different manners; as a rule, the special case of $x = 1$ (one line) will be found easier to handle than the general case. For instance, a differential equation can be derived,

$$\binom{x}{0} f(z) - \binom{x}{1} f'(z) + \binom{x}{2} f''(z) - \dots \binom{x}{x} f^{(x)}(z) = f(z - t);$$

$f(z) = 0$ for all negative values of z being known in advance, it is possible, by integration of the above, to determine the variations of the function, first from $z = 0$ to $z = t$, then from $z = t$ to $z = 2t$, *&c.* The determination of the integration constants, however, will cause difficulties; everything works out smoothly only in the special case of $x = 1$, as further described in my article in "Nyt Tidsskrift for Matematik", 1909, where this case is treated in the indicated manner.

Instead, an integral equation may be employed, *viz.*

$$f(z) = \int_0^\infty f(z + u - t) \frac{u^{x-1}}{(x-1)!} e^{-u} du$$

which immediately leads to a (sometimes) rather convenient numerical solution, but hardly to an explicit mathematical solution.

In the following we shall move along a quite different path, beginning with the introduction of a set of constants: $a_0, a_1, a_2, \dots a_{x-1}$; these are functions of y, or, if you like, of α, α denoting the ratio of y to x. These constants are determined, as we shall see, by inference from some elementary considerations leading to the employment of certain infinite series, all the terms of which are values of the function $e^{-y} \cdot \frac{y^x}{x!}$, and in a tabular representation of the function, the terms of each series will be placed along one or another oblique line, and distributed at equal intervals. *K. Pearson's* collection of tables contains such a table, although for positive values of y only; a similar table comprising negative values of y is given below in the appendix[1]). It should further be noted, with respect to the function $e^{-y} \cdot \frac{y^x}{x!}$, that its significance for the present problem, and for several other ones as well, depends on the following important theorem, the mathematical contents of which was found by *Poisson:* The probability of an arbitrary number of calls (x) being originated during an interval of time with an average number of calls y,

[1]) This table is omitted in the present reprint, as it is identical with Table 2, p. 137, to which the reader is referred.

is equal to $e^{-y}\frac{y^x}{x!}$. I give a simple proof of this theorem in the appendix below.

In some cases, the determination of the constants $a_0, a_1, a_2 \ldots a_{x-1}$ in the manner indicated is very useful; in other cases, such as when α is great (nearly 1), it is very unpractical, however, as the series then are slowly convergent. We shall therefore also give the determination of the constants in a different and more aesthetic form. We introduce a set of auxiliary terms, most often imaginary, $\beta, \gamma, \ldots$, their total number being x when α is included; they are determined by means of a certain transcendental equation in which they are roots. By using a theorem set forth by Mr. *J. L. W. V. Jensen*, Telephone Engineer-in-Chief, Ph. D., the infinite series mentioned can then be summed. The solution of our problem will then appear in a simple and convenient form.

In the following I shall pass in view the two special cases explicitly and uniformly, first $x = 1$, *i. e.* 1 line (in sections 2—6), then $x = 2$, *i. e.* 2 lines (in sections 7 — 11); consequently, I have considered it unnecessary to account for the proof of the general case expressly.

2. *The simpler case of $x = 1$; definition of a_0.*

We understand by a_0 the probability that there will be no waiting time after an arbitrary call. Here we have immediately $a_0 = 1 - \alpha$, a_0 being the probability that the line is unoccupied, and α being the probability that it is occupied.

3. *The table and the oblique lines.*

When x and y both are variable, the table of the Poisson function $e^{-y}\frac{y^x}{x!}$ will fill a plane; we may begin with placing an x-axis and a y-axis in the plane (*e. g.* the x-axis downwards, and the y-axis pointing to the right), and then inscribe each separate value of the function as near as possible to the point determined by the coordinates x and y. Incidentally, we shall have to deal with integral values of x only; and if desired, the negative values of x can be omitted, the function here being 0. Now, we imagine a certain set of oblique lines being laid in the plane, all having the directional coefficient α. On each line we select a number of equally spaced points, each interval corresponding to an increase of 1 in the abscissa, and of α in the ordinate. The sum of the functional values under consideration is denoted by the letter σ, to which is added as indices the coordinates of one of the points, the situation of all the other points being also given hereby. If this point is situated on the x axis, however, the second index — which is 0 — may be omitted, for the sake of brevity. We permit these series to go on infinitely in both directions or, if you choose, in the one direction, and in the other direction until the terms automatically become equal to 0. If only those of the terms corresponding to points with positive ordinates be included in the series, the sum is denoted by s; if only the other terms, *viz.* those corresponding to points with negative ordinates (and 0), be included, the sum is denoted by r. In both cases are added indices, as previously mentioned. Thus, we have always

$$r_{x,y} + s_{x,y} = \sigma_{x,y}.$$

In many cases σ and s are identical and $r = 0$, *viz.* when the oblique line intersects the negative part of the x-axis. The convergence of the series is easily realized.

4. Relations concerning a_0.

Regardless of the fact that we have already found the value of a_0 it will now be useful to prove the following relations:—

$$\left.\begin{aligned} a_0 &= 1 - a_0 s_0 \\ 0 &= 1 - a_0 s_{-1}, \end{aligned}\right\} \qquad (1\text{—}2)$$

where, in accordance with the foregoing,

$$\left.\begin{aligned} s_0 &= e^{-a}\frac{a^1}{1!} + e^{-2a}\frac{(2a)^2}{2!} + \ldots \\ s_{-1} &= e^{-a}\frac{a^0}{0!} + e^{-2a}\frac{(2a)^1}{1!} + \ldots \end{aligned}\right\} \qquad (3\text{—}4)$$

As we shall see later, the two equations (1—2) can be given a different form by introducing the sums σ instead of the sums s; but we will prove them first in the above form.

The equation (1) can be proved as follows: By considering in detail all the cases where an arbitrary call suffers a waiting time, it will be seen that the cases can be distributed, or arranged in groups, thus:—

1) During the preceding time interval of duration t (or a) there was 1 call

2) - - - - - - - $2t$ - were 2 calls

3) - - - - - - - $3t$ - - 3 calls,

etc.

An infinite number of groups is obtained; considering, however, that the probabilities in question form a convergent series, there can be no doubt that the aggregate probability $1 - a_0$ sought-after really exists in the form of a certain limit value; a similar remark could be made at several points in the following. Care should be taken, in the arranging in groups mentioned, that no case be placed under two different groups; to avoid uncertainty in this respect we will decide upon always preferring the group with the higher number to that with the lower number. Agreement with this is found in that, in group no. 1 above, 1 call is stated (*i. e.*, just one call, and no more), and the following groups are in analogy with this; but the cases which, accordingly, should be included must now be sifted further. It is easily seen that the probability that a case really belongs under the group where it has been placed temporarily, is identical with the probability that an arbitrarily chosen call will not have to suffer a waiting time; in other words, it is equal to a_0. For, if we suppose that a case has been put, temporarily, under (*e. g.*) group 3, then we know that there were 3 calls during the preceding time interval of $3\,t$; but that is all we know. We must then take the point of time that is $3\,t$ previous to the call and, from there, seek further back in time; first an interval t, to see whether 1 call can be found here; then an interval $2\,t$, to see whether 2 calls can be found here; &c. We must, thus, undertake the same investigation — although starting from a different point of time — as when we recently began enumerating the cases leading to a waiting time. — Accordingly, we get

$$a_0 = 1 - a_0\left(e^{-a}\frac{a^1}{1!} + e^{-2a}\frac{(2\,a)^2}{2!} + \ldots\right)$$

or, shorter, $$a_0 = 1 - a_0 s_0, \qquad q.\ e.\ d.$$

The equation (2) can be proved in a quite similar manner; we shall not dwell on that, however, as equation (1) strictly speaking will suffice. By inserting σ instead of s, the appearance of equations (1) and (2) becomes simpler and more uniform, *viz.*

$$\left.\begin{aligned} 1 &= a_0\sigma_0 \\ 1 &= a_0\sigma_{-1}. \end{aligned}\right\} \qquad (5—6)$$

The significance of these two equations (their number could easily be increased) is, for the present, that a_0 can be found by means of either of them (we leave out of account that we have already found a_0 in a simpler way here where $x = 1$). But they are, as a matter of fact, significant in another respect also, which will be dealt with later.

5. The summation of the infinite series.

The infinite series σ, as employed in the above, can be summed by means of a theorem by *Jensen* (Acta mathematica XXVI, 1902, p. 309, formula 7). With slightly altered denotations, the theorem reads:

$$\frac{1}{1-\alpha} = e^{-a} \cdot \frac{a^0}{0!} + e^{-(a+\alpha)} \cdot \frac{(a+\alpha)^1}{1!} + e^{-(a+2\alpha)} \cdot \frac{(a+2\alpha)^2}{2!} + \ldots, \quad (7)$$

and it is valid for all values (real and imaginary) of a when only $|\alpha e^{-\alpha}| < \frac{1}{e}$, and also $|\alpha| < 1$. It is valid, at any rate, for the values of α we are using here, *viz.* the positive numbers between 0 and 1. Just now we shall consider 2 special cases only: $a = 0$ and $a = \alpha$. Then we have

$$\left.\begin{aligned} \sigma_0 &= e^{-0}\frac{0^0}{0!} + e^{-\alpha}\frac{\alpha^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^2}{2!} + \ldots = \frac{1}{1-\alpha} \\ \sigma_{-1} &= e^{-\alpha}\frac{\alpha^0}{0!} + e^{-2\alpha}\frac{(2\alpha)^1}{1!} + \ldots = \frac{1}{1-\alpha}. \end{aligned}\right\} \quad (8\text{—}9)$$

Using (8—9) and (5—6) we find that $a_0 = 1 - \alpha$ which we knew already. Simple expressions can also be found for the quantities s, although not quite so simple as in the case of σ.

6. The application of a_0 to the solution of the main problem.

We will now find $S\left(\frac{>}{z}\right)$, *i. e.* the probability of a waiting time greater than z, or its complement $S\left(\frac{\leq}{z}\right)$. For this purpose, we return to the equation (1) which we shall now proceed to generalize. On the left-hand side we substitute $S\left(\frac{\leq}{z}\right)$ for a_0, and on the right, $s_{(0,-z)}$ for s_0; in other words, we move the oblique line concerned a step z to the left. The equation thus obtained,

$$\left.\begin{aligned} S\left(\frac{\leq}{z}\right) &= 1 - a_0 \cdot s_{(0,-z)} \\ \text{or} \qquad S\left(\frac{>}{z}\right) &= a_0 \cdot s_{(0,-z)} \end{aligned}\right\} \quad (10)$$

is proved in quite the same manner as the original equation (1). Also the equation (2) can be generalized in a similar way, but we need not go into that.

The equation (10) has the drawback of containing an infinite series which, however, can be easily replaced with a finite series. We have

$$r_{(0,-z)} + s_{(0,-z)} = \sigma_{(0,-z)} \tag{11}$$

$$a_0 \cdot \sigma_{(0,-z)} = 1, \tag{12}$$

the latter resulting from *Jensen's* theorem.

By means of this, we get from (10)

$$S\left(\frac{>}{z}\right) = 1 - a_0 r_{(0,-z)} \tag{13}$$

or

$$S\left(\frac{<}{z}\right) = \quad a_0 r_{(0,-z)}. \tag{14}$$

This formula is valid for all values of z, but the number of terms resulting depends on whether we are dealing with first interval, $0 < z < t$, or second interval, $t < z < 2t$, &c. As I have done elsewhere, certain special constants b_0, b_1; c_0, c_1, c_2, c_3; &c., can here be used to write the formulae concerning each separate interval, but these constants are easily derivable from a_0. As a matter of fact, the formula (14) expresses everything in the simplest and most convenient form.

7. The case of $x = 2$; definition of a_1 and a_0.

We understand by a_1 the probability that there will be no waiting time after an arbitrarily chosen call (or that there will be at least one unoccupied line); by a_0 we understand the probability that there will be no waiting time after a call when there has been another call immediately preceding it (or that a random call will find both the lines concerned unoccupied). We get directly the relation,

$$a_1 + a_0 = 2(1 - a); \tag{15}$$

for, a_1 is the probability that there will be at least 1 line unoccupied at at any arbitrarily chosen moment, and a_0 is the probability that there will be 2 unoccupied lines; and $2(1 - a)$ is the average number of unoccupied lines. — A number of equations sufficient for the determination of a_1 and a_0 will be given later.

8. The table and the oblique lines.

Here, too, we use the previously mentioned table and define certain sums, partly finite, partly infinite, and denoted by the letters σ, r, and s; and we attach the definitions to certain oblique lines having the directional coefficient α and being situated in the plane of the table. The only distinction is that the difference in abscissa for the successive points selected along an oblique line is not 1, but 2; the difference in ordinate is not α, but 2α. As before, we use two indices, *viz.* the abscissa and ordinate for one of the points; the ordinate, however, can be omitted when equal to 0. Additional distinctive marks consisting of 1 vertical stroke, respectively 2 vertical strokes are prefixed in the cases where there is a risk of mistaking the previously defined sums for those now introduced. We have also here

$$r_{(x,\,y)} + s_{(x,\,y)} = \sigma_{(x,\,y)}, \tag{16}$$

where the symbol σ indicates the inclusion of all terms (or all which are not 0); s, on the other hand, indicates the inclusion of those only which correspond to points with positive ordinates; and r, that only those corresponding to negative ordinates, and 0, are included. — (σ and s are equal and $r = 0$ in many cases, *viz.* when the oblique line intersects the negative part of the x axis, or possibly the positive part between the points $x = 0$ and $x = 1$.

9. Determination of a_0 and a_1.

We will prove that

$$\left.\begin{aligned} a_1 &= 1 - (a_1 s_0 + a_0 s_1) \\ a_0 &= 1 - (a_1 s_{-1} + a_0 s_0) \\ 0 &= 1 - (a_1 s_{-2} + a_0 s_{-1}), \end{aligned}\right\} \tag{17—19}$$

where, in accordance with the foregoing,

$$\left.\begin{aligned} s_1 &= e^{-t}\frac{t^3}{3!} + e^{-2t}\frac{(2t)^5}{5!} + \ldots \\ s_0 &= e^{-t}\frac{t^2}{2!} + e^{-2t}\frac{(2t)^4}{4!} + \ldots \\ s_{-1} &= e^{-t}\frac{t^1}{1!} + e^{-2t}\frac{(2t)^3}{3!} + \ldots \\ s_{-2} &= e^{-t}\frac{t^0}{0!} + e^{-2t}\frac{(2t)^2}{2!} + \ldots \end{aligned}\right\} \tag{20—23}$$

or, if you like,

$$\left.\begin{aligned} s_1 &= e^{-2\alpha}\frac{(2\alpha)^3}{3!} + e^{-4\alpha}\frac{(4\alpha)^5}{5!} + \ldots. \\ s_0 &= e^{-2\alpha}\frac{(2\alpha)^2}{2!} + e^{-4\alpha}\frac{(4\alpha)^4}{4!} + \ldots. \\ s_{-1} &= e^{-2\alpha}\frac{(2\alpha)^1}{1!} + e^{-4\alpha}\frac{(4\alpha)^3}{3!} + \ldots. \\ s_{-2} &= e^{-2\alpha}\frac{(2\alpha)^0}{0!} + e^{-4\alpha}\frac{(4\alpha)^2}{2!} + \ldots. \end{aligned}\right\} \quad (24\text{—}27)$$

As we shall see later, the equations (17—19) can be expressed in a different form by introducing the sums σ instead of the sums s; but we will prove them in the form as given above.

The equation (17) can be proved as follows: If we consider all the cases where an arbitrary call must suffer a waiting time, it will be evident that these cases can be arranged in various groups, such as:—

1) During the preceding time interval of the duration t (or 2α) there were 2 or 3 calls
2) - - - - - - - - $2t$ there were 4 or 5 calls
3) - - - - - - - - $3t$ there were 6 or 7 calls, and so on.

Care should be taken, however, that no one case be placed under two different groups; to avoid uncertainty in this respect we will decide upon always preferring the group with the higher number to that with the lower number. Agreement with this is found in that, in group no. 1 above, the specification reads "2 or 3 calls" (*i. e.* and no more), and the following groups are in analogy with this; but the cases which, accordingly, should be included must now be sifted further. It will be necessary to distinguish, within group no. 1, between a subordinate group a (2 calls) and a subordinate group b (3 calls), and similarly within the other groups. It is now easy to see that the probability that a case really belongs under a subordinate group a where it has temporarily been placed, is identical with the probability that an arbitrarily chosen call will not have to suffer a waiting time; in other words, a_1. Likewise, the probability that a case temporarily placed under a subordinate group b really belongs there, is the same as the probability that an arbitrary call will not have to wait and furthermore finds both lines unoccupied; in other words, a_0. Hence we have

$$a_1 = 1 - a_1\left(e^{-t}\frac{t^2}{2!} + e^{-2t}\frac{(2t)^4}{4!} + \ldots\right) - a_0\left(e^{-t}\frac{t^3}{3!} + e^{-2t}\frac{(2t)^5}{5!} + \ldots\right)$$

or

$$a_1 = 1 - (a_1 s_0 + a_0 s_1), \qquad q.\ e.\ d.$$

The equations (18) and (19) can be proved in a similar manner, but we shall not dwell on that.

By inserting σ instead of s, the equations (17—19) become simpler and more uniform, *viz.*

$$\left.\begin{aligned} 1 &= a_1\sigma_0 + a_0\sigma_1 \\ 1 &= a_1\sigma_{-1} + a_0\sigma_0 \\ 1 &= a_1\sigma_{-2} + a_0\sigma_{-1} \end{aligned}\right\} \qquad (28\text{—}30)$$

The significance of these three equations (their number could easily be increased) is that the constants a_1 and a_0 can be determined by means of any two of them (or by any one of them when the equation (15) is utilized). Incidentally, they are also significant in another respect which we shall see later.

10. Introduction of the new constant β, summation of the infinite series, and determination of a_1 and a_0.

The infinite series, in the summation of which we are now interested, are the following:

$$\left.\begin{aligned} \sigma_1 &= e^0\frac{0^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^3}{3!} + e^{-4\alpha}\frac{(4\alpha)^5}{5!} + \ldots \\ \sigma_0 &= e^0\frac{0^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^2}{2!} + e^{-4\alpha}\frac{(4\alpha)^4}{4!} + \ldots \\ \sigma_{-1} &= e^{-2\alpha}\frac{(2\alpha)^1}{1!} + e^{-4\alpha}\frac{(4\alpha)^3}{3!} + \ldots \\ \sigma_{-2} &= e^{-2\alpha}\frac{(2\alpha)^0}{0!} + e^{-4\alpha}\frac{(4\alpha)^2}{2!} + \ldots \end{aligned}\right\} \qquad (31\text{—}34)$$

We know the sums of the following series which are closely related to those just mentioned (*Jensen*'s theorem, equation no. 7 above):

$$\left.\begin{aligned}
|\sigma_1 &= e^{a}\frac{(-a)^0}{0!} + e^0\frac{0^1}{1!} + e^{-a}\frac{a^2}{2!} + e^{-2a}\frac{(2a)^3}{3!} + \dots = \frac{1}{1-a}\\
|\sigma_0 &= e^{0}\frac{0^0}{0!} + e^{-a}\frac{a^1}{1!} + e^{-2a}\frac{(2a)^2}{2!} + e^{-3a}\frac{(3a)^3}{3!} + \dots = \frac{1}{1-a}\\
|\sigma_{-1} &= e^{-a}\frac{a^0}{0!} + e^{-2a}\frac{(2a)^1}{1!} + e^{-3a}\frac{(3a)^2}{2!} + e^{-4a}\frac{(4a)^3}{3!} + \dots = \frac{1}{1-a}\\
|\sigma_{-2} &= e^{-2a}\frac{(2a)^0}{0!} + e^{-3a}\frac{(3a)^1}{1!} + e^{-4a}\frac{(4a)^2}{2!} + e^{-5a}\frac{(5a)^3}{3!} + \dots = \frac{1}{1-a}
\end{aligned}\right\}(35-38)$$

The series $\|\sigma$ and $|\sigma$ only differ in that every second of the terms contained in the latter is missing in the former. Now, we obtain from the equations (35—38):

$$\left.\begin{aligned}
\frac{1}{a}\cdot|\sigma_1 &= (ae^{-a})^{-1}\cdot\frac{(-1)^0}{0!} + (ae^{-a})^0\frac{0^1}{1!} + (ae^{-a})^1\frac{1^2}{2!} + \dots = \frac{1}{a(1-a)}\\
|\sigma_0 &= (ae^{-a})^0\frac{0^0}{0!} + (ae^{-a})^1\frac{1^1}{1!} + (ae^{-a})^2\frac{2^2}{2!} + \dots = \frac{1}{1-a}\\
a\cdot|\sigma_{-1} &= (ae^{-a})^1\frac{1^0}{0!} + (ae^{-a})^2\frac{2^1}{1!} + (ae^{-a})^3\frac{3^2}{2!} + \dots = \frac{a}{1-a}\\
a^2\cdot|\sigma_{-2} &= (ae^{-a})^2\frac{2^0}{0!} + (ae^{-a})^3\frac{3^1}{1!} + (ae^{-a})^4\frac{4^2}{2!} + \dots = \frac{a^2}{1-a}
\end{aligned}\right\}(39-42)$$

A scheme of obtaining the values of the four quantities

$$\frac{1}{a}\cdot\|\sigma_1,\quad \|\sigma_0,\quad a\cdot\|\sigma_{-1},\quad a^2\cdot\|\sigma_{-2},$$

by removing every second term (*viz.* those with odd exponents) from the four series above, all of which are arranged according to the powers of ae^{-a}, can be put in practice in a convenient way by replacing a in each series with a new constant β, as given by the equation

$$\beta e^{-\beta} = -ae^{-a}$$

and then taking the mean value of the old and the new result. The equation will always have one, and only one, serviceable (negative) root (*i. e.* one to which *Jensen's* theorem can be applied).

Thus, we get

$$\left.\begin{aligned}
\frac{1}{\alpha}\cdot \Vert\sigma_1 &= \frac{1}{2}\left(\frac{1}{\alpha(1-\alpha)}+\frac{1}{\beta(1-\beta)}\right)\\
\Vert\sigma_0 &= \frac{1}{2}\left(\frac{1}{1-\alpha}+\frac{1}{1-\beta}\right)\\
\alpha\cdot \Vert\sigma_{-1} &= \frac{1}{2}\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\right)\\
\alpha^2\cdot \Vert\sigma_{-2} &= \frac{1}{2}\left(\frac{\alpha^2}{1-\alpha}+\frac{\beta^2}{1-\beta}\right)
\end{aligned}\right\} \quad (43-45)$$

or,

$$\left.\begin{aligned}
\Vert\sigma_1 &= \frac{\alpha}{2}\left(\frac{1}{\alpha(1-\alpha)}+\frac{1}{\beta(1-\beta)}\right)\\
\Vert\sigma_0 &= \frac{1}{2}\left(\frac{1}{1-\alpha}+\frac{1}{1-\beta}\right)\\
\Vert\sigma_{-1} &= \frac{1}{2\alpha}\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\right)\\
\Vert\sigma_{-2} &= \frac{1}{2\alpha^2}\left(\frac{\alpha^2}{1-\alpha}+\frac{\beta^2}{1-\beta}\right)
\end{aligned}\right\} \quad (47-50)$$

It is possible, of course, to find the quantities $\Vert s$ just as we have found the quantities $\Vert \sigma$ (expressed in terms of α and β), but the expressions will not be quite so neat. — From the equations (28—30) and (47—50) we now obtain

$$\left.\begin{aligned}
a_1 &= 2(1-\alpha)\frac{\alpha}{\alpha-\beta}\\
a_0 &= -2(1-\alpha)\frac{\beta}{\alpha-\beta}
\end{aligned}\right\} \quad (51-52)$$

which, by insertion, will satisfy not only (28—30), but also all those analogous to the latter, *i. e.* more generally the equation

$$a_1\sigma_p + a_0\sigma_{p+1} = 1;$$

for we get

$$2(1-\alpha)\frac{\alpha}{\alpha-\beta}\cdot\frac{\alpha^p}{2}\left(\frac{1}{\alpha^p(1-\alpha)}+\frac{1}{\beta^p(1-\beta)}\right)$$

$$-2(1-\alpha)\frac{\beta}{\alpha-\beta}\cdot\frac{\alpha^{p+1}}{2}\left(\frac{1}{\alpha^{p+1}(1-\alpha)}+\frac{1}{\beta^{p+1}(1-\beta)}\right)=1$$

11. Applying the quantities found, a_1 and a_0, to the solution of the main problem.

We shall now determine $S\binom{>}{z}$, *i. e.* the probability of a waiting time greater than z; or its complement $S\binom{\leq}{z}$. We consider the equation (17) which we will now generalize. On the left side we replace a_1 with $S\binom{\leq}{z}$, and on the right s_0 and s_1 with $s_{(0,-z)}$ and $s_{(1,-z)}$, respectively; in other words, we move the oblique line concerned a step z to the left. The equation thus obtained,

$$S\binom{\leq}{z} = 1 - (a_1 \cdot s_{(0,\,-z)} + a_0 \cdot s_{(1,\,-z)}) \tag{53}$$

or

$$S\binom{>}{z} = a_1 \cdot s_{(0,\,-z)} + a_0 \cdot s_{(1,\,-z)} \tag{54}$$

can be proved in quite the same manner as the equation (17). Also (18) and (19) can be generalized in a similar way, but we need not go into that.

Now, the infinite series in the equations (53—54) can be replaced with finite ones. We have

$$r_{(0,\,-z)} + s_{(0,\,-z)} = \sigma_{(0,\,-z)} \tag{55}$$

$$r_{(1,\,-z)} + s_{(1,\,-z)} = \sigma_{(1,\,-z)} \tag{56}$$

$$a_1\sigma_{(0,\,-z)} + a_0\sigma_{(1,\,-z)} = 1; \tag{57}$$

hence

$$S\binom{>}{z} = 1 - a_1 r_{(0,\,-z)} - a_0 r_{(1,\,-z)} \tag{58}$$

or

$$S\binom{\leq}{z} = a_1 r_{(0,\,-z)} + a_0 r_{(1,\,-z)} \tag{59}$$

The formula is valid for all values of z. The number of terms contained in the formula depends on whether z belongs in the first interval $0 < z < t$, or in the second interval $t < z < 2\,t$, and so on.

It is easy to write out, as I have done elsewhere, the special formulae valid for the separate intervals; the constants involved here, b_0, b_1, b_2, b_3; $c_0, c_1, c_2, c_3, c_4, c_5$; &c., are easily derived from a_0 and a_1. However, the formula (58—59) really expresses everything, and perhaps even in the very best form, at that.

12. Appendix.

The proof of the theorem used in the above, *viz.:* When, during a given time, the average number of calls is y, the probability of x calls being originated will be

$$S_x = e^{-y} \frac{y^x}{x!}.$$

Let it be assumed that the time in consideration represents a portion of a very long time over which a correspondingly great number of calls is dispersed so that y calls, at an average, fall within the time portion considered. The duration of the latter can be called y, the unit of time being chosen in such a manner as to give an average of 1 call per unit of time. Let us suppose that, in a certain case, say, 5 calls occur within the time y, and let us move y a short distance dy; then there will be a probability $\frac{5\,dy}{y}$ that 1 of the 5 calls is shut out so that the number is reduced to 4. *Vice versa*, if we had 4 calls before y was moved, there will be a probability $\frac{5\,dy}{y}$ of gaining 1 new call by the movement. But the transitions from 5 to 4 and *vice versa* must neutralize each other, and so

$$S_5 \cdot \frac{5}{y} = S_4.$$

This result — and analogous results — give us the ratio between the successive members of the sequence $S_0, S_1, S_2, \ldots$; these must then be proportional to

$$1, \quad \frac{y}{1!}, \quad \frac{y^2}{2!}, \quad \frac{y^3}{3!}, \ldots$$

As, necessarily,

$$S_0 + S_1 + S_2 + \ldots = 1,$$

and as

$$1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots = e^y,$$

we obtain

$$S_0 = e^{-y}, \quad S_1 = e^{-y} \frac{y}{1!}, \quad S_2 = e^{-y} \frac{y^2}{2!} \ldots,$$

q. e. d.

Table 1.

Values of the constants a_0 for $x = 1$; a_1 and a_0 for $x = 2$.

	$x = 1$	$x = 2$	
a	a_0	a_1	a_0
0·0	1·0	1·000000	1·000000
1	0·9	0·982234	0·817766
2	8	935507	664493
3	7	866418	533582
4	6	779180	420820
5	5	676741	323259
6	4	561316	238684
7	3	434633	165367
8	2	298104	101896
9	1	152892	047108
1·0	0	000000	000000

Table 2.

Values of $S\left(\frac{\leq}{z}\right)$ for $x = 1$ and $x = 2$.

$x = 1.$

z / a	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
0·0	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1	0·900	0·995	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
2	800	884	0·977	0·991	0·998	0·999	1·000	1·000	1·000	1·000	1·000
3	700	774	855	945	967	983	0·992	0·996	0·998	0·999	0·999
4	600	663	733	810	895	923	947	965	977	984	990
5	500	553	611	675	746	824	856	885	910	931	947
6	400	442	489	540	597	659	729	761	792	822	849
7	300	332	366	405	448	495	547	605	635	665	694
8	200	221	244	270	298	330	364	403	445	470	495
9	100	111	122	135	149	165	182	201	223	246	261
1·0	000	000	000	000	000	000	000	000	000	000	000

$x = 2.$

a \ z	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
0·0	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1	0·982	0·995	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
2	936	960	0·980	0·994	0·999	1·000	1·000	1·000	1·000	1·000	1·000
3	866	899	928	953	974	0·989	0·995	0·997	0·999	0·999	1·000
4	779	815	849	881	911	938	960	976	985	989	0·992
5	677	712	748	783	817	849	880	909	931	949	961
6	561	594	627	661	695	729	762	794	824	852	877
7	435	462	490	520	550	580	611	642	673	703	732
8	298	318	339	361	384	407	432	457	482	508	533
9	153	164	175	187	200	213	227	241	256	272	288
1·0	000	000	000	000	000	000	000	000	000	000	000

4. THE APPLICATION OF THE THEORY OF PROBABILITIES IN TELEPHONE ADMINISTRATION

From the Scandinavian H. C. Örsted Congress in Copenhagen, 1920; a lecture.

1. The problem forming the principal subject of this lecture may temporarily be defined as follows: — We have a certain amount of telephone traffic, *i. e.* a certain number of calls per unit of time, and we have a certain number of lines to take the conversations in question, or else a certain number of operators to establish the desired connexions. What result may we expect, on the assumption that any one line (or operator) can always be substituted for any other line (or operator) if the latter happens to be occupied at the moment? The circumstances are not always the same; sometimes the systems are arranged in such a way that an incoming call will be lost if the whole group is occupied, in which case the problem is to determine the number of lost calls. But there are also systems arranged so as to provide for a delay in answering, and then the problem is to find the probability of delays, or waiting times, of certain durations, including the average waiting time. Experience has taught something about these matters, though not enough. From time to time, new systems are introduced, and new propositions made; and it would be much too costly to subject them all to practical experiments. Therefore it is necessary to theorize, and this applies to all branches of telephony (manual, automatic, &c.): the problems differ to some extent, but they are based upon the same things, and the circumstances are analogous in many respects.

I shall now make a few historical remarks concerning, especially, the oldest works in this line. Here, a short essay by Mr. *F. Johannsen*, director of the Telephone Company of Copenhagen, deserves to be mentioned first. The essay deals with waiting times, particularly in manual exchanges; it was first printed in 1907, then in "Ingeniøren", 1910, and in the "Post Office E. E. Journal" 1910—1911. The solution of the problem contained herein was very simple — not exact, but serviceable for the time being; incidentally, Mr. Johannsen caused a new and more thorough investigation of the rather difficult problem — and other similar problems — to be undertaken. A considerable interest had now been taken in the matter, in Denmark as well as abroad, and people set to work upon it. As regards automatic exchanges, the first important work was done in 1913 by Mr. *P. V. Christensen*, Engineer-in-Chief to the Telephone Company of Copenhagen, his subject being the calculation of the number of selectors (or lines) in all the stages of an automatic system; he managed to illuminate this question in all essentials, at least with respect to the systems that were in use at the time. *Grinsted* in England, and Dr. *Spiecker* in Germany should also be mentioned. Among the later authors whose works undoubtedly are fresh in the memory may be mentioned: Mr. *Engset*, department manager; Dr. *R. Holm;* and Dr. *Lely*. The most prominent of the mathematicians who by their works have supplied the basis of the Telephone Theory, are *Poisson* (1837, Recherces sur la probabilité &c.) and our contemporary, Dr. *J. L. W. V. Jensen*, Engineer-in-Chief to the Telephone Co. of Copenhagen (a dissertation in Acta Mathematica, 1902, dedicated to the late *N. H. Abel*).

It will be necessary, unfortunately, to omit from the following all the proofs of the various results stated therein, so for the proofs I shall have to refer to my printed works[1]), where a good many more numerical results than I am able to give here, can also be found.

2. On the Distribution in Time of the Calls, and the Holding Time.

a. There is a very important formula concerning the distribution of calls. We let y be the average number of calls during a certain interval of time, and we want to find the probability that 0, 1, 2,.... calls will actually be originated. These probabilities will be, respectively,

$$P_0 = e^{-y}, \quad P_1 = e^{-y} \cdot y, \quad P_2 = e^{-y} \frac{y^2}{2!} \ldots,$$

or, generally,

$$P_x = e^{-y} \frac{y^x}{x!}.$$

It will often be convenient to choose the unit of time so that the average number of calls per unit of time is equal to 1; the above proposition can then be expressed somewhat more briefly as y is simply the length of the interval of time. This formula is strictly exact if the calls are distributed or dispersed quite accidentally, any one call being independent of any other, over a very long time, of which the interval under consideration constitutes a small part. In practice, the formula can be applied without hesitation to the calls, but not to the conversations as these are not fundamentally independent of each other, but competing. From a mathematical point of view the formula is due to Poisson; it has been found again several times later, and it is being utilized more and more in telephony — thus, in Denmark from 1909, in England from 1907 (published

[1]) See pp. 131—171.

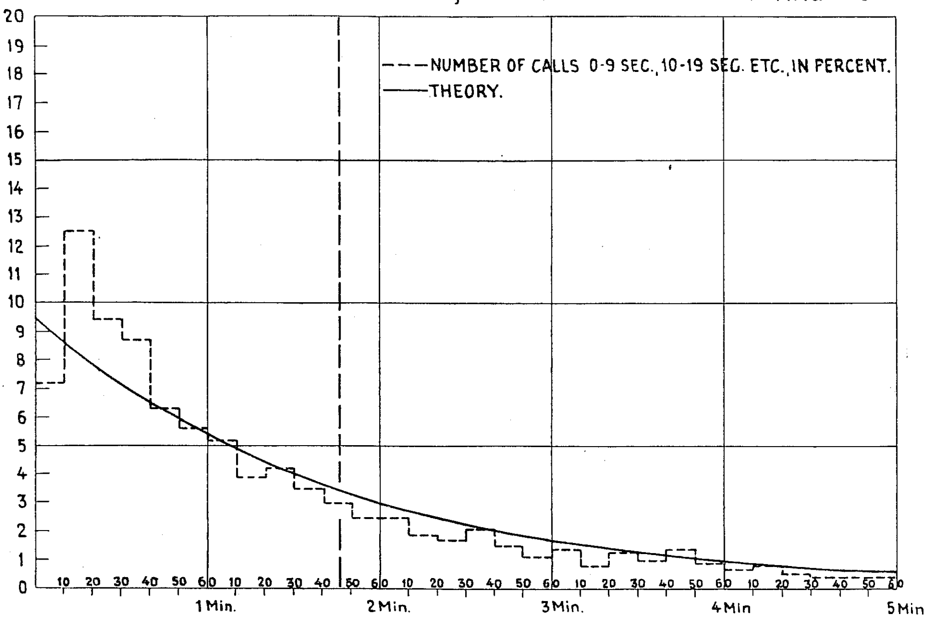
1915) — and it is important in other, quite different domains also. The function $e^{-y}\frac{y^x}{x!}$ has been tabulated, more or less extensively, by *Bortkiewics*, *H. E. Soper*, and *R. Holm*. Incidentally, by means of such a table it is possible to derive a new table — by successive additions — of the quantities

$$P_0, \quad P_0 + P_1, \quad P_0 + P_1 + P_2, \; \ldots ,$$

the significance of which is obvious.

Table 1.

DURATION OF 2461 CONVERSATIONS, COPENHAGEN MAIN EXCHANGE 1916



b. The duration of a conversation, or the holding time, has formerly been regarded as being constant, as a rule. This is rather correct in the case of trunk calls which are cut off at the expiration of a certain period; it is also fairly correct as to the operator's conversations with the subscribers (while the calls are being operated), if not in all exchanges. For the most important class of conversations, *i. e.* the ordinary local calls, is valid with good approximation the law of distribution

$$S = e^{-n}, \quad \text{whence} \quad S' = - e^{-n},$$

where S denotes the probability that the duration n will be exceeded; the unit of time is here supposed to be identical with the average duration.

The two equations express that the probability of a current conversation's being on the verge of termination is independent of the time it has lasted already — which might reasonably have been expected beforehand. The main point is, however, that experience agrees nicely with the law of distribution, *cf.* table 1 where experiences from the main exchange of Copenhagen are represented graphically. Also Dr. *Lely*'s results seem to agree with it in the essentials. A larger number of laws of distribution might be desirable, for the sake of eventualities; so, quite naturally, we arrive at a series T_1, T_2, T_3 ...., the equations of which are given in the table below together with the corresponding values of the standard deviation (dispersion).

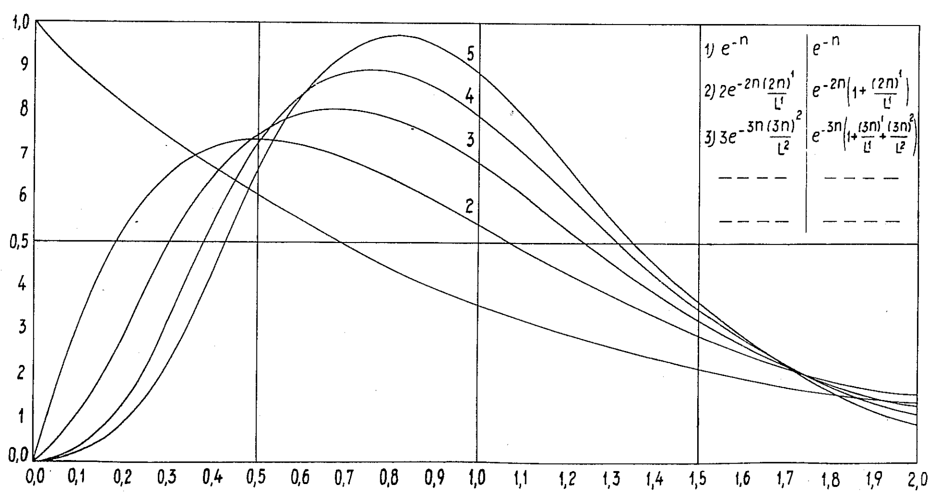
T_1	$S = e^{-n}$	$S' = -e^{-n}$	1
T_2	$S = e^{-2n}\left(1 + \frac{2n}{1!}\right)$	$S' = -2e^{-2n}\frac{2n}{1!}$	$\frac{1}{\sqrt{2}}$
T_3	$S = e^{-3n}\left(1 + \frac{3n}{1!} + \frac{(3n)^2}{2!}\right)$	$S' = -3e^{-3n}\frac{(3n)^2}{2!}$	$\frac{1}{\sqrt{3}}$
....	....	....	......
....	....	....	......

The two previously mentioned laws of distribution are thus included in the table, the latter as T_1 (top), the former as T_∞ (bottom) (constant duration,

Table 2.

DURATION OF A TELEPHONE CONVERSATION:

DIFFERENT POSSIBLE LAWS OF DISTRIBUTION.



dispersion zero), a number of intermediate equations being intercalated between the two extreme laws. T_2 was previously suggested by *Lely*. The curves in table 2 will serve to illustrate what has been said here.

It should be noticed that these different hypotheses, when applied to the solution of problems, in many cases will lead to the same result; in other cases the results will be more or less different. In order to avoid complications it will be advantageous to employ, as a rule, and if possible either T_1 or T_∞ which, each in its own way, are simpler than the others.

3. *On Obstruction or Barred Access ("Systems Without Waiting Arrangement")*.

a. We suppose that there are x lines between two points A and B, and that the intensity of traffic is y. The term "intensity of traffic" means the average number of calls being originated in the course of a time that is equal to the average holding time. The ratio of y to x can be used instead of y; this ratio is called the a of the lines, or the occupation. Putting B equal to the probability of obstruction or barred access, *i. e.* the probability of finding all the x lines occupied, we obtain:

$$B = \frac{\frac{y^x}{x!}}{1 + \frac{y}{1!} + \ldots + \frac{y^x}{x!}}.$$

There are various approximative formulae for this quantity which is often called the grade of service; but then, the exact formula is very simple. When calculating in practice the factor e^{-y} should be applied both in numerator and denominator as this will make it possible to use the numerical tables mentioned in the preceding section. In the collection of formulae, table 3, are furthermore given the probabilities that 0, 1, 2, ... lines are occupied. It is possible to prove that these results and those following later in this section are independent of the law of distribution that applies to the durations of calls. The proof is based on the theory of "statistical equilibrium" which I shall not try to explain here. Nearly all of the following formulae are based on the same theory. The curve chart, table 4, shows graphic representations, partly of B as a function of y for $x = 4, 5, 6$, and partly, for comparison, of the results of a special kind of experiments based upon 100 imaginary calls distributed as "accidentally" as possible; I found the "times on" by taking the last 3 figures from each of the 100 7-figure logarithms log 1 — log 100 and arranging the 100 numbers thus produced in the order of their magnitudes. Choosing first a certain holding time, it is quite easy to find out what happens to the 100 calls; next, another holding time is chosen (*i. e.* a different y), and so on.

b. The next, somewhat more difficult problem to be considered is this: — We have x lines between 2 points A and B; also, n lines connecting A with n different points $C_1, C_2, \ldots. C_n$. The traffic between B and these points C proceeds over A; the intensity of traffic is β for each channel, or βn altogether, and whether a call originates from this or that side is of no importance in this connexion. Several questions may now be asked; but especially the following three: — What is the probability g_n that one particular line out of the n lines is available for a call? What is the probability g_x that one, or more than one, of the x cooperating lines is available for a call? What is the probability g_{nx} of all the lines' being available? Once these probabilities have been found, we can derive others directly from them, such as:

$$g_n - g_{n,x} \quad \text{and} \quad \frac{g_n - g_{n,x}}{g_n},$$

the significance of which is obvious. (*Cf.* the collection of formulae). The problem mentioned has previously been treated by *T. Engset* who employed an approximative method, however, for which reason his results differ a little from mine.

c. Systems with "Grading and Interconnecting" ("Mischung und Staffelung").

These systems play an important part in automatic exchanges, and much interest is taken, in *e. g.* England, U. S. A., and Germany, in the question of how much can be achieved by their employment. The fact is that automatic selectors have only a limited — usually not very large — number of contacts, and so give admission only to just that number of lines. Let us suppose that we have a total of x lines at our disposal, the selectors, however, having only k contacts; then, in each separate case only k out of the x lines will be hunted through in search of a disengaged line. It is a well known fact by now that it is not a happy solution to divide the lines into a number of separate groups, each containing k, and divide the total traffic y among the latter. Another procedure is to divide the lines into a number of separate groups and one common group, the latter to be used only when the separate group concerned has been hunted through and found busy. Or a circular permutation may be used, *i. e.* a division of the traffic into x parts, the first of which hunts the lines $1, 2, 3, \ldots. k$; the second $2, 3, 4, \ldots. (k + 1)$; and so on all the way. In this manner all the lines will be used equally much. Still, this method

Table 3. Formulae concerning Loss.

$$B = \frac{\frac{y^x}{x!}}{1+\frac{y}{1!}+\cdots+\frac{y^x}{x!}}$$

$$\begin{cases} S_0 = \dfrac{1}{1+\frac{y}{1!}+\cdots+\frac{y^x}{x!}} \\ S_1 = \dfrac{y}{1+\frac{y}{1!}+\cdots+\frac{y^x}{x!}} \\ \cdots\cdots\cdots\cdots \\ S_x = \dfrac{\frac{y^x}{x!}}{1+\frac{y}{1!}+\cdots+\frac{y^x}{x!}} \end{cases}$$

$$B = \frac{P_x}{P_0+P_1+\cdots+P_x}$$

$$\begin{cases} P_0 = e^{-y} \\ P_1 = e^{-y}\dfrac{y}{1!} \\ \cdots\cdots\cdots \\ P_x = e^{-y}\dfrac{y^x}{x!} \end{cases}$$

$$g_n = \frac{\binom{n-1}{0}+\binom{n-1}{1}\beta+\binom{n-1}{2}\beta^2+\ldots+\binom{n-1}{x}\beta^x}{\binom{n}{0}+\binom{n}{1}\beta+\binom{n}{2}\beta^2+\ldots+\binom{n}{x}\beta^x}$$

$$g_x = \frac{\binom{n}{0}+\binom{n}{1}\beta+\binom{n}{2}\beta^2+\ldots+\binom{n}{x-1}\beta^{x-1}}{\binom{n}{0}+\binom{n}{1}\beta+\binom{n}{2}\beta^2+\ldots+\binom{n}{x}\beta^x}$$

$$g_{n,\,x} = \frac{\binom{n-1}{0}+\binom{n-1}{1}\beta+\binom{n-1}{2}\beta^2+\ldots+\binom{n-1}{x-1}\beta^{x-1}}{\binom{n}{0}+\binom{n}{1}\beta+\binom{n}{2}\beta^2+\ldots+\binom{n}{x}\beta^x}$$

$$g_n - g_{n,\,x} = \frac{\binom{n-1}{x}\beta^x}{\binom{n}{0}+\binom{n}{1}\beta+\binom{n}{2}\beta^2+\ldots+\binom{n}{x}\beta^x}$$

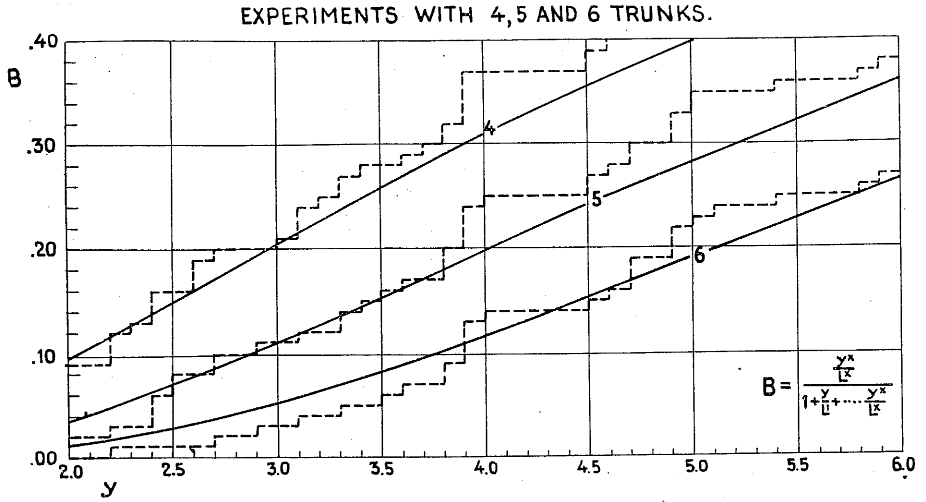
$$\frac{g_n - g_{n,\,x}}{g_n} = \frac{\binom{n-1}{x}\beta^x}{\binom{n-1}{0}+\binom{n-1}{1}\beta+\binom{n-1}{2}\beta^2+\ldots+\binom{n-1}{x}\beta^x}$$

Table 3 (continued). Formula concerning Loss in Grading Systems.

$$B = \frac{T_0 P_0 + T_1 P_1 + \ldots + T_x P_x}{N_0 P_0 + N_1 P_1 + \ldots + N_x P_x}$$

Index.	P.	N.	T.
0	e^{-y}	1	0
1	$e^{-y}\frac{y}{1!}$	1	0
2	$e^{-y}\frac{y^2}{2!}$	1	0
⋮	⋮	⋮	⋮
k	$e^{-y}\frac{y_k}{k!}$	1	$N_k - N_{k+1}$
$k+1$	$e^{-y}\frac{y^{k+1}}{(k+1)!}$	$1 - \frac{\binom{x}{x}}{\binom{x}{x-k}}$	$N_{k+1} - N_{k+2}$
$k+2$	$e^{-y}\frac{y^{k+2}}{(k+2)!}$	$\left(1 - \frac{\binom{x}{x}}{\binom{x}{x-k}}\right)\left(1 - \frac{\binom{x+1}{x}}{\binom{x+1}{x-k}}\right)$	$N_{k+2} - N_{k+3}$
$k+3$	$e^{-y}\frac{y^{k+3}}{(k+3)!}$	$\left(1 - \frac{\binom{x}{x}}{\binom{x}{x-k}}\right)\left(1 - \frac{\binom{x+1}{x}}{\binom{x+1}{x-k}}\right)\left(1 - \frac{\binom{x+2}{x}}{\binom{x+2}{x-k}}\right)$	$N_{k+3} - N_{k+4}$
⋮	⋮	⋮	⋮
x	$e^{-y}\frac{y^x}{x!}$	$\left(1 - \frac{\binom{x}{x}}{\binom{x}{x-k}}\right) \cdots \left(1 - \frac{\binom{2x-k-1}{x}}{\binom{2x-k-1}{x-k}}\right)$	N_x

Table 4.



is not the best possible, *i. e.* the one that gives the traffic the best chance of getting through. The ideal method consists in the k lines being chosen in all possible ways (*i. e.* not only in the recently mentioned x ways) and, furthermore, subjected to hunting at random — not in a predetermined order. This necessitates, of course, a division of the traffic into a large number of portions. Under these presuppositions we arrive at the formulae given in table 3, and the numerical results (for $x = 5, 10, 15, 20, \dots$ and $k = 5, 10, 15, 20, \dots$) in table 5; only a few of these numerical values have been published before. To make the calculations easier, I give here a couple of approximative formulae; they can often replace the exact formulae, although they must be used with caution:—

$$B = \left(\frac{y}{x}\right)^k \quad \dots \text{(for great values of } y \text{ and } x\text{)}$$

$$B = y^k \frac{(x-k)!}{x!} = e^{-y} \cdot \frac{y^x}{x!} : \left(e^{-y} \frac{y^{x-k}}{(x-k)!}\right) \dots \text{(for small values of } y\text{)}$$

There are probably several arrangements that are more or less different from the above mentioned ideal method, but nevertheless give almost as good results. For the time being, however, no precise information on this point is available.

Table 5. Loss, in Thousandth Parts, for x Lines among which k are chosen.

$x =$	5		10			15				20					25						30							18			
$k =$	0	5	0	5	10	0	5	10	15	0	5	10	15	20	0	5	10	15	20	25	0	5	10	15	20	25	30	0	5	10	15
$y = 0$																															
0.5																															
1		3																													
1.5		14																													
2		37			0																										
2.5		70			0																										
3		110			1		1																								
3.5		154			2		1																								
4		199			5		3																								
4.5		243		32	10		5																								
5	1000	285	1000	47	18	1000			0																						
5.5		324		64	29																									0	
6		360		83	43		17	2	1																					0	
6.5		394		104	60																									1	
7		425		126	79		30	6	3																					1	
7.5		453		149	100																									2	
8		479		173	122		50	14	9		14	1	0	0																3	
8.5		503		196	145						19	2	1	0																5	
9		525		219	168		75	29	20		24	3	1	1																7	
9.5		545		242	191						30	5	2	1																11	
10	1000	564	1000	264	215	1000	103	48	36	1000	37	7	3	2	1000	14	1	0			1000									15	
10.5					237						44	9	4	3																20	
11					260		133	74	59		52	13	6	5		21	1	0	0											27	
11.5					282		149	88	72		61	17	9	7																34	
12					302		164	102	86		70	23	12	10		30	3	1	1	0		13	0							43	
12.5					328		180	118	100		80	29	17	14																	
13					341		196	133	116		91	36	22	18		41	6	2	1	1		18	1								
13.5					360		211	149	132		102	44	28	24																	
14					377		227	166	148		113	53	35	30		53	11	5	3	2		25	2	0	0						
14.5					394		242	182	164		125	62	43	37																	
15					410	1000	258	198	180	1000	136	72	52	46	1000	68	18	9	6	5	1000	33	4	1	0	0	0				

Table 5 (continued).

$x =$	5		10			15				20					25						30							18			
$k =$	0	5	0	5	10	0	5	10	15	0	5	10	15	20	0	5	10	15	20	25	0	5	10	15	20	25	30	0	5	10	15
$y =$																															
15.5							272	214	197			82	62	55																	
16							286	230	213			93	73	64		84	27	15	11	9		43	6	2	1	1	1				
16.5							300	246	228			105	83	75																	
17							314	260	224			117	94	86		101	39	23	18	16		54	10	4	2	2	1				
17.5							326	275	259			129	106	97																	
18							340	290	274			141	118	109		120	53	35	28	25		66	15	7	4	3	3				
18.5									289					121																	
19														134		138	69	48	40	36		79	22	11	7	6	5				
19.5																															
20															1000	158	86	64	54	50	1000	93	31	17	12	9	8				
20.5																															
21																						108	41	24	18	15	14				
21.5																															
22																						124	54	34	26	22	21				
22.5																															
23																								45	36	31	29				
23.5																															
24																								57	47	43	40				
24.5																															
25																										55	53				
25.5																															
26																											67				
26.5																															
27																											82				
27.5																															
28																											98				
28.5																															
29																											115				
29.5																															
30																											132				

Table 6. Formulae concerning Delays. (Hypothesis T_∞).

$x = 1$ $\quad S(<) = a_0 \cdot \bar{r}(0, -z)$

$(z = yn = \alpha n)$ $a_0 = 1 - \alpha$ $\quad\Big|\quad$ $\bar{r}(0, -z) = e^z + e^{z-\alpha} \dfrac{(\alpha - z)^1}{1!} + \dots$

$$M = \frac{\alpha^2}{2\alpha(1-\alpha)} = \frac{\alpha}{2(1-\alpha)}$$

$x = 2$ $\quad S(<) = a_1 \cdot \bar{\bar{r}}(0, -z) + a_0 \cdot \bar{\bar{r}}(1, -z)$

$(z = yn = 2\alpha n)$

$$\begin{cases} a_1 = 2(1-\alpha)\dfrac{\alpha}{\alpha-\beta} & \bar{\bar{r}}(0,-z) = e^z + e^{z-2\alpha}\dfrac{(2\alpha - z)^2}{2!} + \dots \\ a_0 = -2(1-\alpha)\dfrac{\beta}{\alpha-\beta} & \bar{\bar{r}}(1,-z) = e^z \dfrac{z^1}{1!} + e^{z-2\alpha}\dfrac{(2\alpha - z)^3}{3!} + \dots \end{cases}$$

$$\beta e^{-\beta} = -\alpha e^{-\alpha}$$

$$M = \frac{1}{2(\alpha - \beta)} + \frac{2\alpha^2 - 1}{4\alpha(1-\alpha)}$$

$x = 3$ $\quad S(<) = a_2 \cdot \bar{\bar{\bar{r}}}(0, -z) + a_1 \cdot \bar{\bar{\bar{r}}}(1, -z) + a_0 \cdot \bar{\bar{\bar{r}}}(2, -z)$

$(z = yn = 3\alpha n)$

$$\begin{cases} a_2 = 3(1-\alpha)\dfrac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} & \bar{\bar{\bar{r}}}(0,-z) = e^z + e^{z-3\alpha}\dfrac{(3\alpha - z)^3}{3!} + \dots \\ a_1 = -3(1-\alpha)\dfrac{\alpha(\beta+\gamma)}{(\alpha-\beta)(\alpha-\gamma)} & \bar{\bar{\bar{r}}}(1,-z) = e^z \dfrac{z^1}{1!} + e^{z-3\alpha}\dfrac{(3\alpha - z)^4}{4!} + \dots \\ a_0 = 3(1-\alpha)\dfrac{\beta\gamma}{(\alpha-\beta)(\alpha-\gamma)} & \bar{\bar{\bar{r}}}(2,-z) = e^z \dfrac{z^2}{2!} + e^{z-3\alpha}\dfrac{(3\alpha - z)^5}{5!} + \dots \end{cases}$$

$$\begin{cases} \beta e^{-\beta} = \alpha e^{-\alpha} \cdot k \\ \gamma e^{-\gamma} = \alpha e^{-\alpha} \cdot k^2 \end{cases}$$

$$k = \sqrt[3]{1}$$

$$M = \frac{1}{3(\alpha - \beta)} + \frac{1}{3(\alpha - \gamma)} + \frac{3\alpha^2 - 2}{6\alpha(1-\alpha)}$$

Table 6 (continued). Formulae concerning Delays. (Hypothesis T_1.)

$x = 1$	$S(>) = c \cdot e^{-(1-y)n}$ $c = y$ $M = \frac{c}{1-y} = \frac{a}{1-a}$
$x = 2$	$S(>) = c \cdot e^{-(2-y)n}$ $c = \frac{\frac{y^2}{2}}{1+\frac{y}{2}}$ $M = \frac{c}{2-y}$
$x = 3$	$S(>) = c \cdot e^{-(3-y)n}$ $c = \frac{\frac{y^3}{6}}{1+\frac{2}{3}y+\frac{y^2}{6}}$ $M = \frac{c}{3-y}$
x arbitrary	$S(>) = c \cdot e^{-(x-y)n}$ $c = \frac{\frac{y^x}{x!}}{\left(1+\ldots+\frac{y^x}{x!}\right)-\frac{y}{x}\left(1+\ldots+\frac{y^{x-1}}{(x-1)!}\right)}$ or $c = \frac{\frac{y^x}{x!}}{\left(1+\ldots+\frac{y^{x-1}}{(x-1)!}\right)-\frac{y}{x}\left(1+\ldots+\frac{y^{x-2}}{(x-2)!}\right)}$ $M = \frac{c}{x-y}$

Table 6 (continued). Appendix. Some Different Hypotheses (for $x = 1$ only)

T_1	T_2	T_3	T_∞
$u = \alpha$	$u = \frac{1}{2}\alpha$	$u = \frac{1}{3}\alpha$	
$r - u = 0$	$r^2 - ur - u = 0$	$r^3 - ur^2 - ur - u = 0$	
$c_1 = 1 - \alpha$	$\begin{cases} c_1 + c_2 = 1 - \alpha \\ c_1 r_1 + c_2 r_2 = \frac{1}{2}\alpha(1-\alpha) \end{cases}$	$\begin{cases} c_1 + c_2 + c_3 = 1 - \alpha \\ c_1 r_1 + c_2 r_2 + c_3 r_3 = \frac{1}{3}\alpha(1-\alpha) \\ c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2 = \frac{1}{3}\alpha(1-\alpha)(1+\frac{1}{3}\alpha) \end{cases}$	See overleaf
$S = \frac{c_1 r_1}{1-r_1} e^{(r_1-1)n}$ $= \alpha e^{-(1-\alpha)n}$	$S = \frac{c_1 r_1}{1-r_1} e^{2(r_1-1)n} + \frac{c_2 r_2}{1-r_2} e^{2(r_2-1)n}$	$S = \frac{c_1 r_1}{1-r_1} e^{3(r_1-1)n} + \frac{c_2 r_2}{1-r_2} e^{3(r_2-1)n} + \frac{c_3 r_3}{1-r_3} e^{3(r_3-1)n}$	
$M = \frac{2}{2} \cdot \frac{\alpha}{1-\alpha} = \frac{\alpha}{1-\alpha}$	$M = \frac{3}{4} \cdot \frac{\alpha}{1-\alpha}$	$M = \frac{4}{6} \cdot \frac{\alpha}{1-\alpha} = \frac{2}{3} \cdot \frac{\alpha}{1-\alpha}$	$M = \frac{1}{2} \cdot \frac{\alpha}{1-\alpha}$

4. Delays in Answering (Waiting Times).

a. Let us temporarily suppose that the conversations are of constant duration ("hypothesis T_∞"). The number of lines is x, and the intensity of traffic is y; we must have $y < x$. Now, in the first place, we have to determine $S\ (>)$, the probability that the delay will exceed a certain length of time; or $S\ (<)$, the probability that the delay will be below a certain length of time. This length of time is called n when the holding time is used as the unit of time, but z when the time during which an average of 1 call is originated, is used as unit of time. We prefer here to use the holding time; we have also $z = y \cdot n$. — The necessary formulae are contained in the collection of formulae, table 6; some of the final results are shown graphically in tables 7, 8, and 9, to which, however, must be added a few explanatory remarks. The numerical calculation is based upon a table of the Poisson function already mentioned several times:

$$e^{-y} \frac{y^x}{x!},$$

as calculated this time for *negative* values of the variable, y; for, the main part of the work consists in calculating series of the following type:

$$\bar{r}\,(0, -z) = e^z + e^{z-a} \frac{(a-z)^1}{1!} + e^{z-2a} \frac{(2a-z)^2}{2!} + \ldots$$

for $x = 1$, or

$$\bar{\bar{r}}\,(0, -z) = e^z + e^{z-2a} \frac{(2a-z)^2}{2!} + e^{z-4a} \frac{(4a-z)^4}{4!} + \ldots$$

$$\bar{\bar{r}}\,(1, -z) = e^z \cdot \frac{z^1}{1!} + e^{z-2a} \frac{(2a-z)^3}{3!} + e^{z-4a} \frac{(4a-z)^5}{5!} + \ldots$$

for $x = 2$, &c. Obviously, all the terms of these series are examples of the Poisson function and can be found in the table just mentioned. It is emphasized that these series are not infinite; they continue only as far as is given by the table. Otherwise, the number of terms to be included will depend upon z (or n, if you like), and transition takes place for

$$n = 0, 1, 2, 3, \ldots .$$

That is why *e. g.* the curves in table 7 exhibit bends at $n = 1$.

The reader will perhaps notice the new, simple formulae for the average delay; they are due to Mr. *H. C. Nybølle,* M. A., in collaboration with the author.

b. We will now leave the hypothesis T_∞ and pass on to the second main case T_1. Here, the formulae are simpler, the curves are more smooth, all calculations are easier. The results will be found to differ to some extent from those previously mentioned; thus, the average delay will be 2 times greater for $x = 1$; $1^1/_2$ to 2 times greater for $x = 2$; and $1^1/_3$ to 2 times greater for $x = 3$, the smaller numbers corresponding to small values of y and the greater to great values of y.

c. Finally some formulae are included showing, though only for $x = 1$, the results corresponding to the hypotheses $T_1, T_2, \ldots, T_\infty$. The curve chart, table 10, gives an idea of the results obtained from the formulae when $\alpha = 0.2$.

Table 7.

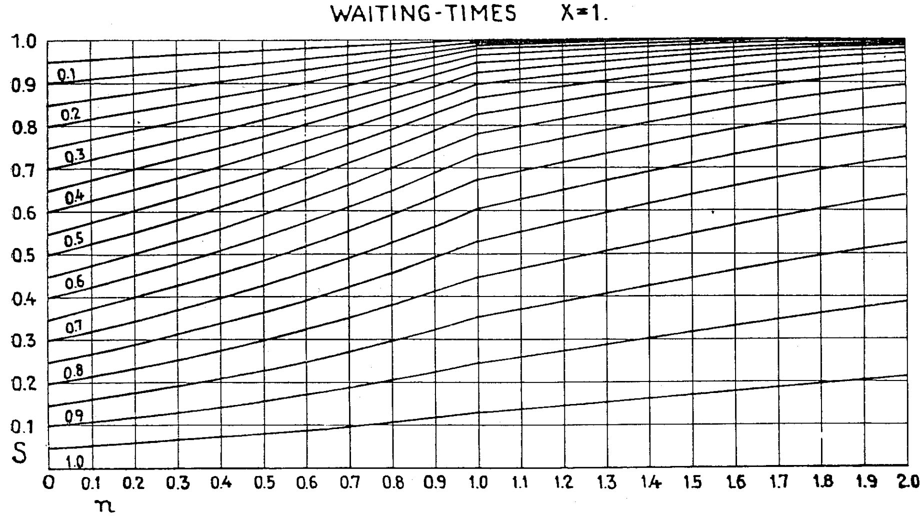


Table 8.

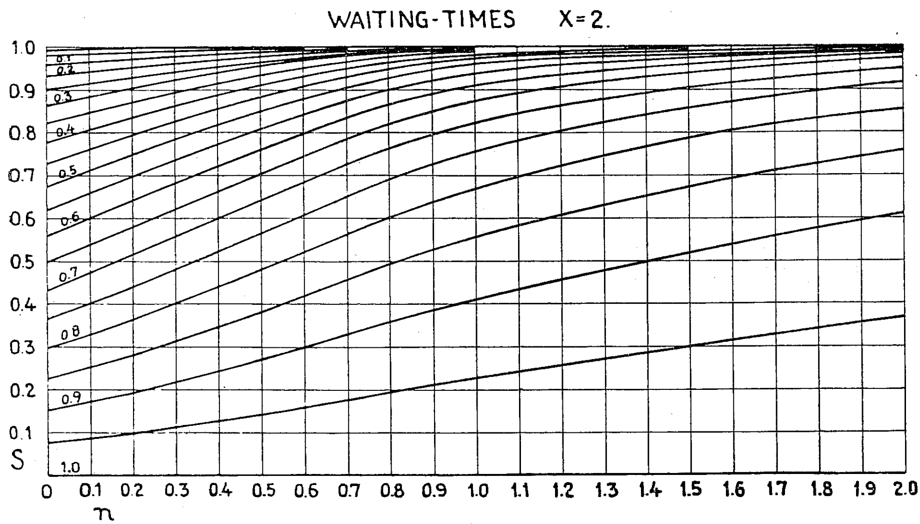


Table 9.

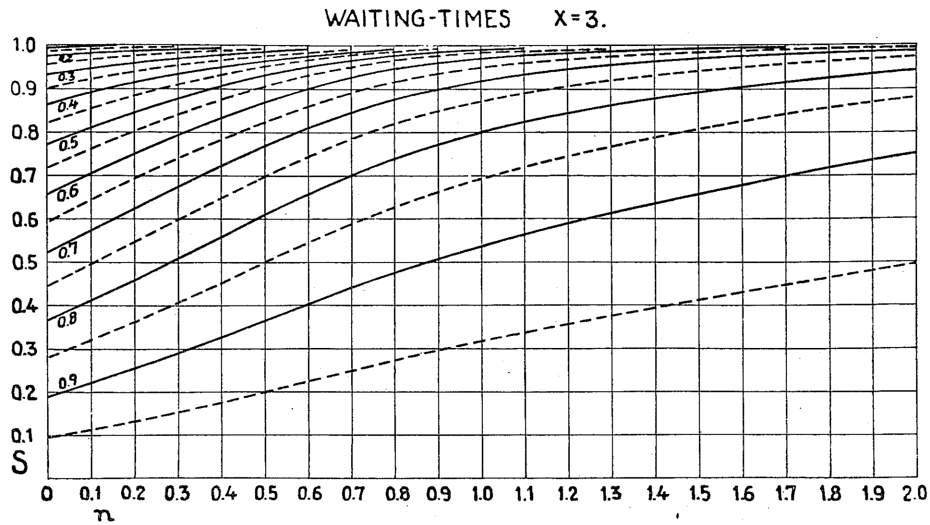


Table 10.

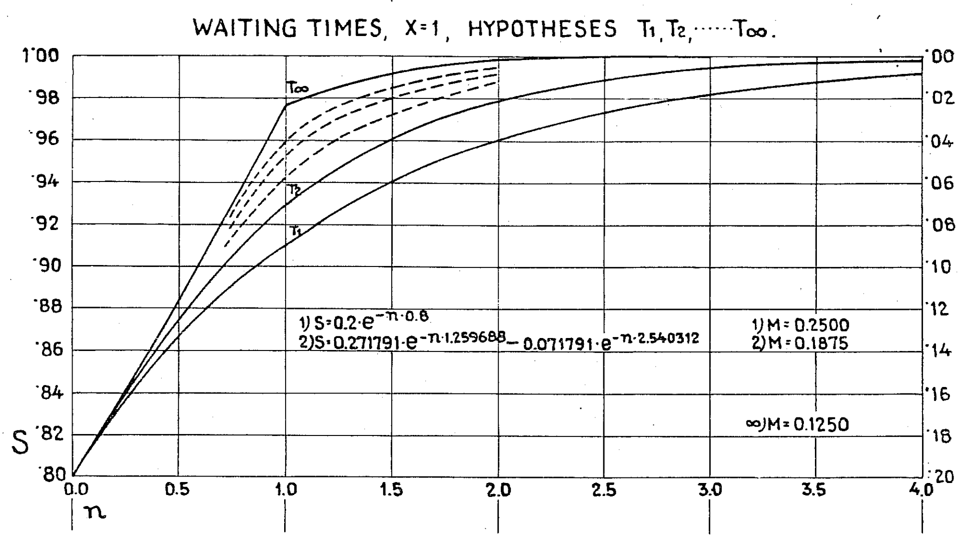


Table 11. Collection of Formulae.
Different systems and rules for transferring and delays.

	$x = 1$
$V = 0$ (Full transfer)	$\frac{(1)}{(0)} = \alpha + \beta$ $(0) + (1) = 1$ $S = 0$ $M = 0$ $B = (1)$
$V = 1$ (Regular transfer)	$\frac{(1)}{(0)} = \alpha + \beta;\ \frac{(2)}{(1)} = \alpha$ $(0) + (1) + (2) = 1$ $S = (1) \cdot e^{-n}$ $M = (1)$ $B = (2)$
$V = 2$ (Reduced transfer)	$\frac{(1)}{(0)} = \alpha + \beta;\ \frac{(2)}{(1)} = \alpha;\ \frac{(3)}{(2)} = \alpha$ $(0) + (1) + (2) + (3) = 1$ $S = (1) \cdot e^{-n} + (2) \cdot e^{-n} \cdot \left(1 + \frac{n}{1!}\right)$ $M = (1) \cdot 1 + (2) \cdot 2$ $B = (3)$
$V = 3$ (Extra reduced transfer)	$\frac{(1)}{(0)} = \alpha + \beta;\ \frac{(2)}{(1)} = \alpha;\ \frac{(3)}{(2)} = \alpha;\ \frac{(4)}{(3)} = \alpha$ $(0) + (1) + (2) + (3) + (4) = 1$ $S = (1) \cdot e^{-n} + (2) \cdot e^{-n} \cdot \left(1 + \frac{n}{1!}\right) + (3) \cdot e^{-n} \cdot \left(1 + \frac{n}{1!} + \frac{n^2}{2!}\right)$ $M = (1) \cdot 1 + (2) \cdot 2 + (3) \cdot 3$ $B = (4)$
$V = \infty$ (No transfer)	$\frac{(1)}{(0)} = \alpha;\ \frac{(2)}{(1)} = \alpha;\ \frac{(3)}{(2)} = \alpha \ldots$ $(0) + (1) + (2) + (3) + \ldots = 1$ $S = (1) \cdot e^{-n} + (2) \cdot e^{-n} \cdot \left(1 + \frac{n}{1!}\right) + (3) \cdot e^{-n}\left(1 + \frac{n}{1!} + \frac{n^2}{2!}\right) + \ldots$ or $S = \frac{(1)}{1-\alpha} \cdot e^{-n(1-\alpha)} = \alpha \cdot e^{-n(1-\alpha)}$ $M = (1) \cdot 1 + (2) \cdot 2 + (3) \cdot 3 + \ldots$ or $M = \frac{(1)}{(1-\alpha)^2} = \frac{\alpha}{1-\alpha}$

$$\beta = B \cdot \frac{\alpha}{1-\alpha}$$

Table 11 (continued).

$x = 2$
$\frac{(1)}{(0)} = 2\,\alpha + 2\,\beta;\quad \frac{(2)}{(1)} = (2\,\alpha + \beta) : 2$ $(0) + (1) + (2) = 1$ $S = 0$ $M = 0$ $B = (2)$
$\frac{(1)}{(0)} = 2\,\alpha + 2\,\beta;\quad \frac{(2)}{(1)} = (2\,\alpha + \beta) : 2;\quad \frac{(3)}{(2)} = \alpha$ $(0) + (1) + (2) + (3) = 1$ $S = (2) \cdot e^{-2n}$ $M = (2) \cdot \frac{1}{2}$ $B = (3)$
$\frac{(1)}{(0)} = 2\,\alpha + 2\,\beta;\quad \frac{(2)}{(1)} = (2\,\alpha + \beta) : 2;\quad \frac{(3)}{(2)} = \alpha;\quad \frac{(4)}{(3)} = \alpha$ $(0) + (1) + (2) + (3) + (4) = 1$ $S = (2) \cdot e^{-2n} + (3) \cdot e^{-2n} \cdot \left(1 + \frac{2\,n}{1!}\right)$ $M = (2) \cdot \frac{1}{2} + (3) \cdot \frac{2}{2}$ $B = (4)$
$\frac{(1)}{(0)} = 2\,\alpha + 2\,\beta;\quad \frac{(2)}{(1)} = (2\,\alpha + \beta) : 2;\quad \frac{(3)}{(2)} = \alpha;\quad \frac{(4)}{(3)} = \alpha;\quad \frac{(5)}{(4)} = \alpha$ $(0) + (1) + (2) + (3) + (4) + (5) = 1$ $S = (2) \cdot e^{-2n} + (3) \cdot e^{-2n} \cdot \left(1 + \frac{2\,n}{1!}\right) + (4) \cdot e^{-2n} \left(1 + \frac{2\,n}{1!} + \frac{(2\,n)^2}{2!}\right)$ $M = (2) \cdot \frac{1}{2} + (3) \cdot \frac{2}{2} + (4) \cdot \frac{3}{2}$ $B = (5)$
$\frac{(1)}{(0)} = 2\,\alpha;\quad \frac{(2)}{(1)} = 2\,\alpha : 2;\quad \frac{(3)}{(2)} = \alpha \ldots$ $(0) + (1) + (2) + \ldots = 1$ $S = (2) \cdot e^{-2n} + (3) \cdot e^{-2n} \cdot \left(1 + \frac{2\,n}{1!}\right) + (4) \cdot e^{-2n} \left(1 + \frac{2\,n}{1!} + \frac{(2\,n)^2}{2!}\right) + \ldots$ or $S = \frac{(2)}{1 - \alpha} \cdot e^{-2n(1-\alpha)} = \frac{2\,\alpha^2}{1 + \alpha} \cdot e^{-2n(1-\alpha)}$ $M = (2) \cdot \frac{1}{2} + (3) \cdot \frac{2}{2} + (4) \cdot \frac{3}{2} + \ldots$ or $M = \frac{(2)}{2\,(1 - \alpha)^2} = \frac{\alpha^2}{(1 + \alpha)\,(1 - \alpha)}$

$$\beta = B \cdot \frac{\alpha}{1 - \alpha}$$

Table 11 (continued).

$x = 3$
$\frac{(0)}{(1)} = 3\alpha + 3\beta;\quad \frac{(2)}{(1)} = (3\alpha + 2\beta):2;\quad \frac{(3)}{(2)} = (3\alpha + \beta):3$ $(0) + (1) + (2) + (3) = 1$ $S = 0$ $M = 0$ $B = (3)$
$\frac{(1)}{(0)} = 3\alpha + 3\beta;\quad \frac{(2)}{(1)} = (3\alpha + 2\beta):2;\quad \frac{(3)}{(2)} = (3\alpha + \beta):3;\quad \frac{(4)}{(3)} = \alpha$ $(0) + (1) + (2) + (3) + (4) = 1$ $S = (3) \cdot e^{-3n}$ $M = (3) \cdot \frac{1}{3}$ $B = (4)$
$\frac{(1)}{(0)} = 3\alpha + 3\beta;\quad \frac{(2)}{(1)} = (3\alpha + 2\beta):2;\quad \frac{(3)}{(2)} = (3\alpha + \beta):3;\quad \frac{(4)}{(3)} = \alpha;\quad \frac{(5)}{(4)} = \alpha$ $(0) + (1) + (2) + (3) + (4) + (5) = 1$ $S = (3) \cdot e^{-3n} + (4) \cdot e^{-3n} \cdot \left(1 + \frac{3n}{1!}\right)$ $M = (3) \cdot \frac{1}{3} + (4) \cdot \frac{2}{3}$ $B = (5)$
$\frac{(1)}{(0)} = 3\alpha + 3\beta;\quad \frac{(2)}{(1)} = (3\alpha + 2\beta):2;\quad \frac{(3)}{(2)} = (3\alpha + \beta):3;$ $\frac{(4)}{(3)} = \alpha;\quad \frac{(5)}{(4)} = \alpha;\quad \frac{(6)}{(5)} = \alpha$ $(0) + (1) + (2) + (3) + (4) + (5) + (6) = 1$ $S = (3) \cdot e^{-3n} + (4) \cdot e^{-3n} \cdot \left(1 + \frac{3n}{1!}\right) + (5) \cdot e^{-3n} \cdot \left(1 + \frac{3n}{1!} + \frac{(3n)^2}{2!}\right)$ $M = (3) \cdot \frac{1}{3} + (4) \cdot \frac{2}{3} + (5) \cdot \frac{3}{3}$ $B = (6)$
$\frac{(1)}{(0)} = 3\alpha;\quad \frac{(2)}{(1)} = 3\alpha:2;\quad \frac{(3)}{(2)} = 3\alpha:3;\quad \frac{(4)}{(3)} = \alpha \ldots$ $(0) + (1) + (2) + \ldots = 1$ $S = (3) \cdot e^{-3n} + (4) \cdot e^{-3n} \cdot \left(1 + \frac{3n}{1!}\right) + (5) \cdot e^{-3n} \cdot \left(1 + \frac{3n}{1!} + \frac{(3n)^2}{2!}\right) + \ldots$ or $S = \frac{(3)}{1-\alpha} \cdot e^{-3n(1-\alpha)} = \frac{\frac{9}{2}\alpha^3}{1 + 2\alpha + \frac{3}{2}\alpha^2} \cdot e^{-3n(1-\alpha)}$ $M = (3) \cdot \frac{1}{3} + (4) \cdot \frac{2}{3} + (5) \cdot \frac{3}{3} + \ldots$ or $M = \frac{(3)}{3(1-\alpha)^2} = \frac{\frac{3\alpha^3}{2}}{\left(1 + 2\alpha + \frac{3\alpha^2}{2}\right)(1-\alpha)}$

$$\beta = B \cdot \frac{\alpha}{1-\alpha}$$

Table 12. Formulae concerning Automatic Distribution. (Hypothesis T_1; big groups).

$v_1 = v_2 = \ldots = 0$ (Full transfer)	$\frac{(1)}{(0)} = a_1$ $(0) + (1) = 1$ $S = 0$ $M = 0$ $\frac{a_1}{1 + a_1} = a$
$v_1 = v_2 = \ldots = 1$	$\frac{(1)}{(0)} = a_1 \quad \frac{(2)}{(1)} = a_1$ $(0) + (1) + (2) = 1$ $S = \frac{a_1}{1 + a_1} \cdot e^{-n}$ $M = \frac{a_1}{1 + a_1}$ $\frac{a_1 + a_1^2}{1 + a_1 + a_1^2} = a$
$v_1 = v_2 = \ldots = 2$	$\frac{(1)}{(0)} = a_1 \quad \frac{(2)}{(1)} = a_1 \quad \frac{(3)}{(2)} = a_1$ $(0) + (1) + (2) + (3) = 1$ $S = \frac{a_1 \cdot e^{-n} + a_1^2 \cdot e^{-n}(1 + n)}{1 + a_1 + a_1^2}$ $M = \frac{a_1 + 2a_1^2}{1 + a_1 + a_1^2}$ $\frac{a_1 + a_1^2 + a_1^3}{1 + a_1 + a_1^2 + a_1^3} = a$
$v_1 = v_2 = \ldots = 3$	$\frac{(1)}{(0)} = a_1 \quad \frac{(2)}{(1)} = a_1 \quad \frac{(3)}{(2)} = a_1 \quad \frac{(4)}{(3)} = a_1$ $(0) + (1) + (2) + (3) + (4) = 1$ $S = \frac{a_1 \cdot e^{-n} + a_1^2 \cdot e^{-n}(1+n) + a_1^3 \cdot e^{-n}\left(1 + n + \frac{n^2}{2}\right)}{1 + a_1 + a_1^2 + a_1^3}$ $M = \frac{a_1 + 2a_1^2 + 3a_1^3}{1 + a_1 + a_1^2 + a_1^3}$ $\frac{a_1 + a_1^2 + a_1^3 + a_1^4}{1 + a_1 + a_1^2 + a_1^3 + a_1^4} = a$
$v_1 = [\ldots] = \infty$ (No transfer)	$\frac{(1)}{(0)} = a \quad \frac{(2)}{(1)} = a \ldots$ $(0) + (1) + \ldots = 1$ $S = a \cdot e^{-(1-a)n}$ $M = \frac{a}{1 - a}$ $a_1 = a$

5. Mixed or Compound Problems.

a. We have hitherto considered two main problems corresponding to the two different ways of arranging the systems, either in such a manner that any call will result in a conversation — sometimes after a delay — or in such a manner that such calls as cannot get through immediately, are definitely barred, and consequently lost. Both arrangements are well known. Now it would be interesting to try to compare the inconveniences accompanying either system, on the assumption that the traffic and the number of lines are the same in both cases. A difficulty in this respect is, however, the fact that a delay in answering and a barring of access are heterogeneous quantities. Some information can nevertheless be obtained by tabulating the ratio of the average delay in the one system to the loss in the other, *viz.* that this ratio will increase with the intensity of traffic; in other words, the waiting time system has the advantage when the traffic is feeble, and the other system has it when the traffic is rather greater. But this is what one might have expected in advance.

b. In a large telephone exchange with many operators, a few of these will always, as it were, be "free" when a call arrives; so, in a way, the call *could* be put through immediately if perfect cooperation or distribution of work between the operators could be obtained. No other form of this was known, at one time, than the assistance rendered by neighbours — which, by the way, should not at all be underestimated —; roughly, one might say that there is a number of small groups ($x = 3$, or $x = 2$), within each of which cooperation takes place. Nowadays it is possible, even by several different methods, to get very near to perfection in that every call is directed to a momentarily not too busy operator's position; in this connexion, the concept of "busyness" cannot be defined precisely. It is not advisable to exaggerate in this respect, as a considerable space of time might then easily be wasted on the distribution itself, even if this is carried out more or less automatically. Let us take as an example the system employed in the main exchange of Copenhagen. Here, two different means of distributing the traffic are used. In the first place, the operators can help each other *qua* neighbours, as mentioned above; but they can furthermore get rid of any inopportune call manually and simultaneously with the handling of traffic proper; an automatic selector will then transfer the call to a "free" operator. This transferring should not be carried to excess; in practice, the prescribed procedure is to transfer a new (arriving) call, if the small group of neighbouring operators concerned is fully engaged and 1 call is waiting already. This rule we may denote by $V = 1$; there may be other rules, of course. Apart from this we use the following denotations: The number of calls

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per operator and unit of time is a. The unit of time is equal to the average operating time, and we use the hypothesis T_1 as our basis. The number of directly cooperating operators is x ($x = 1, 2$, or 3). The maximum number of waiting calls is V. All these are the known quantities. B denotes the probability of transfer, and β the number of transferred calls per unoccupied operator during the unit of time. The probability of a calls' being served, or waiting at a group, is denoted by (a); a cannot be greater than $x + V$. The probability of delay beyond a certain limit is called S, and the average delay, M. For the rest, see the collection of formulae (table 11). When employing the formulae, we must proceed as follows: First we choose, by way of experiment, a value of β, then we calculate B, and we go on doing this until the equation

$$\beta = B \cdot \frac{a}{1 - a}$$

is satisfied; this equation expresses that the number of calls transferred *from* one operator's position must equal the number of calls transferred *to* another position. This theory is based upon the supposition that the total number of operators is great; besides, we have not taken into account any time for the perceiving of signals, the transferring of calls, and the movements of the selectors, for which corrections may be introduced if necessary.

c. We will now consider a type of systems in which the distribution is effected quite automatically, without any assistance from the operators. Like the system described under *5b* above, this system has several modifications depending on whether 1, 2, 3, ... calls are permitted to wait in each position. The denotations are, in the essentials, the same as above; a_1 means the number of calls arriving, during the unit of time, in one of the positions not fully loaded. This quantity is evidently somewhat greater than a (*vide* table 12). It is a matter of course that it is presupposed in this theory (as well as in the preceding theory) that the operators' being busy is the only reason for delay — not, *e. g.*, shortage of cords, &c.

It would be easy to enumerate many similar problems which the Theory of Probabilities already is — or in future will be — able to illuminate, thereby contributing to provide for a steadily improving utilization of staff, machinery, and lines. True, there are great difficulties — which fact, however, should not deter anybody. The same applies to many other domains where the application of this science has made better understanding possible, such as: the theory of measuring and counting, the theory of statistics, the science of mortality and genetics, or the theory of molecular motion and collision. In all these fields there are many important questions which have not been answered as yet — or perhaps have not even been formulated yet. This holds good of the theory of telephone traffic, too.

Table 13.

Values of $\frac{y^x}{x!} \cdot e^{-y}$ for $y < 0$[1]).

x	$y = -2{\cdot}1$	$y = -2{\cdot}2$	$y = -2{\cdot}3$	$y = -2{\cdot}4$	$y = -2{\cdot}5$
0	+ 8.166170	+ 9.025013	+ 9.974182	+ 11.023176	+ 12.182494
1	— 17.148957	— 19.855030	— 22.940620	— 26.455623	— 30.456235
2	+ 18.006405	+ 21.840533	+ 26.381713	+ 31.746748	+ 38.070294
3	— 12.604483	— 16.016391	— 20.225980	— 25.397398	— 31.725245
4	+ 6.617354	+ 8.809015	+ 11.629938	+ 15.238439	+ 19.828278
5	— 2.779289	— 3.875967	— 5.349771	— 7.314451	— 9.914139
6	+ 0.972751	+ 1.421188	+ 2.050746	+ 2.925780	+ 4.130891
7	— 0.291825	— 0.446659	— 0.673814	— 1.003125	— 1.475318
8	+ 0.076604	+ 0.122831	+ 0.193722	+ 0.300937	+ 0.461037
9	— 0.017874	— 0.030025	— 0.049507	— 0.080250	— 0.128066
10	+ 0.003754	+ 0.006606	+ 0.011387	+ 0.019260	+ 0.032016
11	— 0.000717	— 0.001321	— 0.002381	— 0.004202	— 0.007276
12	+ 0.000125	+ 0.000242	+ 0.000456	+ 0.000840	+ 0.001516
13	— 0.000020	— 0.000041	— 0.000081	— 0.000155	— 0.000292
14	+ 0.000003	+ 0.000006	+ 0.000013	+ 0.000027	+ 0.000052
15		— 0.000001	— 0.000002	— 0.000004	— 0.000009
16				+ 0.000001	+ 0.000001

x	$y = -2{\cdot}6$	$y = -2{\cdot}7$	$y = -2{\cdot}8$	$y = -2{\cdot}9$	$y = -3{\cdot}0$
0	+ 13.463738	+ 14.879732	+ 16.444647	+ 18.174145	+ 20.085537
1	— 35.005719	— 40.175276	— 46.045011	— 52.705022	— 60.256611
2	+ 45.507434	+ 54.236622	+ 64.463015	+ 76.422281	+ 90.384916
3	— 39.439776	— 48.812960	— 60.165481	— 73.874872	— 90.384916
4	+ 25.635855	+ 32.948748	+ 42.115837	+ 53.559282	+ 67.788687
5	— 13.330644	— 17.792324	— 23.584869	— 31.064384	— 40.673212
6	+ 5.776613	+ 8.006546	+ 11.006272	+ 15.014452	+ 20.336606
7	— 2.145599	— 3.088239	— 4.402509	— 6.220273	— 8.715688
8	+ 0.697320	+ 1.042281	+ 1.540878	+ 2.254849	+ 3.268383
9	— 0.201448	— 0.312684	— 0.479384	— 0.726562	— 1.089461
10	+ 0.052376	+ 0.084425	+ 0.134228	+ 0.210703	+ 0.326838
11	— 0.012380	— 0.020722	— 0.034161	— 0.055549	— 0.089138
12	+ 0.002682	+ 0.004663	+ 0.007971	+ 0.013424	+ 0.022284
13	— 0.000536	— 0.000968	— 0.001717	— 0.002995	— 0.005143
14	+ 0.000100	+ 0.000187	+ 0.000343	+ 0.000620	+ 0.001102
15	— 0.000017	— 0.000034	— 0.000064	— 0.000120	— 0.000220
16	+ 0.000003	+ 0.000006	+ 0.000011	+ 0.000022	+ 0.000041
17		— 0.000001	— 0.000002	— 0.000004	— 0.000007
18				+ 0.000001	+ 0.000001

[1]) Part of this table, comprising values of y ranging from 0.0 to — 2.0, is omitted in the present reprint, as it is identical with Table 2, p. 137, to which the reader is referred.

Table 13.

(continued).

x	$y = - 3{\cdot}1$	$y = - 3{\cdot}2$	$y = - 3{\cdot}3$	$y = - 3{\cdot}4$	$y = - 3{\cdot}5$
0	+ 22.197951	+ 24.532530	+ 27.112639	+ 29.964100	+ 32.115452
1	— 68.813649	— 78.504100	— 89.471708	— 101.877940	— 115.904082
2	+ 106.661156	+ 125.606559	+ 147.628319	+ 173.192498	+ 202.832143
3	— 110.216528	— 133.980330	— 162.391151	— 196.284831	— 236.637500
4	+ 85.417809	+ 107.184264	+ 133.972700	+ 166.842106	+ 207.057813
5	— 52.959042	— 68.597929	— 88.421982	— 113.452632	— 144.940469
6	+ 27.362172	+ 36.585562	+ 48.632090	+ 64.289825	+ 84.548607
7	— 12.117533	— 16.724828	— 22.926557	— 31.226486	— 42.274303
8	+ 4.695544	+ 6.689931	+ 9.457205	+ 13.271257	+ 18.495008
9	— 1.617354	— 2.378642	— 3.467642	— 5.013586	— 7.192503
10	+ 0.501380	+ 0.761166	+ 1.144322	+ 1.704619	+ 2.517376
11	— 0.141298	— 0.221430	— 0.343297	— 0.526882	— 0.800983
12	+ 0.036502	+ 0.059048	+ 0.094407	+ 0.149283	+ 0.233620
13	— 0.008704	— 0.014535	— 0.023965	— 0.039043	— 0.062898
14	+ 0.001927	+ 0.003322	+ 0.005649	+ 0.009482	+ 0.015724
15	— 0.000398	— 0.000709	— 0.001243	— 0.002149	— 0.003669
16	+ 0.000077	+ 0.000142	+ 0.000256	+ 0.000457	+ 0.000803
17	— 0.000014	— 0.000027	— 0.000050	— 0.000091	— 0.000165
18	+ 0.000002	+ 0.000005	+ 0.000009	+ 0.000017	+ 0.000032
19		— 0.000001	— 0.000002	— 0.000003	— 0.000006
20				+ 0.000001	+ 0.000001

x	$y = - 3{\cdot}6$	$y = - 3{\cdot}7$	$y = - 3{\cdot}8$	$y = - 3{\cdot}9$	$y = - 4{\cdot}0$
0	+ 36.598234	+ 40.447304	+ 44.701184	+ 49.402449	+ 54.598150
1	— 131.753644	— 149.555026	— 169.864501	— 192.669552	— 218.392600
2	+ 237.156559	+ 276.861798	+ 322.742552	+ 375.705626	+ 436.785200
3	— 284.587871	— 341.462884	— 408.807233	— 488.417314	— 582.380267
4	+ 256.129084	+ 315.853168	+ 388.366871	+ 476.206881	+ 582.380267
5	— 184.412940	— 233.731344	— 295.158820	— 371.441367	— 465.904214
6	+ 110.647764	+ 144.134329	+ 186.933921	+ 241.436889	+ 310.602809
7	— 56.904564	— 76.185288	— 101.478414	— 134.514838	— 177.487319
8	+ 25.607054	+ 35.235696	+ 48.202246	+ 65.575984	+ 88.743660
9	— 10.242822	— 14.485786	— 20.352060	— 28.416260	— 39.441627
10	+ 3.687416	+ 5.359741	+ 7.733783	+ 11.082341	+ 15.776651
11	— 1.206791	— 1.802822	— 2.671670	— 3.929194	— 5.736964
12	+ 0.362037	+ 0.555870	+ 0.846029	+ 1.276988	+ 1.912321
13	— 0.100256	— 0.158209	— 0.247301	— 0.383096	— 0.588407
14	+ 0.025780	+ 0.041812	+ 0.067125	+ 0.106720	+ 0.168116
15	— 0.006187	— 0.010314	— 0.017005	— 0.027747	— 0.044831
16	+ 0.001392	+ 0.002385	+ 0.004039	+ 0.006763	+ 0.011208
17	— 0.000295	— 0.000519	— 0.000903	— 0.001552	— 0.002637
18	+ 0.000059	+ 0.000107	+ 0.000191	+ 0.000336	+ 0.000586
19	— 0.000011	— 0.000021	— 0.000038	— 0.000069	— 0.000123
20	+ 0.000002	+ 0.000004	+ 0.000007	+ 0.000013	+ 0.000025

Table 14.

Real part ξ of the roots $(\alpha, \beta, \gamma \ldots)$ of the equation $\zeta \cdot e^{-\zeta} = k \cdot a e^{-a}$; $k = \sqrt[x]{1}$. $x = 40$; $a = 0{\cdot}1, 0{\cdot}2 \ldots 0{\cdot}9$.

	0	1/40	2/40	3/40	4/40	5/40	6/40	7/40	8/40	9/40	10/40
a = 0.0	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
0.1	·100000	·098318	·093384	·085523	·075209	·062996	·049459	·035142	·020526	·006026	— ·008014
0.2	·200000	·195248	·181742	·161367	·136390	·108863	·080384	·052096	·024776	— ·001061	— ·025080
0.3	·300000	·289555	·261701	·223504	·180973	·137812	·096012	·056580	·019997	— ·013533	— ·043943
0.4	·400000	·378814	·328822	·269263	·209659	·153432	·101621	·054414	·011697	— ·026738	— ·061112
0.5	·500000	·458594	·379382	·299282	·226074	·160477	·101913	·049651	·003038	— ·038464	— ·075296
0.6	·600000	·522664	·413054	·317103	·234571	·162872	·100015	·044578	— ·004468	— ·047882	— ·086238
0.7	·700000	·566589	·433163	·326920	·238599	·163161	·097680	·040300	— ·010237	— ·054824	— ·094122
0.8	·800000	·591943	·444114	·331982	·240351	·162760	·095759	·037248	— ·014160	— ·059436	— ·099285
0.9	·900000	·604078	·449321	·334305	·241027	·162356	·094585	·035499	— ·016354	— ·061982	— ·102115
a = 1.0	1·000000	·607501	·450801	·334952	·241195	·162206	·094210	·034954	— ·017031	— ·062763	— ·102980

	10/40	11/40	12/40	13/40	14/40	15/40	16/40	17/40	18/40	19/40	20/40
a = 0.0	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
0.1	— ·008014	— ·021317	— ·033660	— ·044866	— ·054799	— ·063353	— ·070447	— ·076020	— ·080031	— ·082448	— ·083256
0.2	— ·025080	— ·047066	— ·066883	— ·084448	— ·099711	— ·112645	— ·123233	— ·131470	— ·137354	— ·140885	— ·142061
0.3	— ·043943	— ·071239	— ·095462	— ·116665	— ·134909	— ·150246	— ·162726	— ·172388	— ·179266	— ·183383	— ·184755
0.4	— ·061112	— ·091635	— ·118496	— ·141854	— ·161844	— ·178581	— ·192157	— ·202642	— ·210094	— ·214550	— ·216032
0.5	— ·075296	— ·107806	— ·136281	— ·160954	— ·182010	— ·199597	— ·213839	— ·224825	— ·232625	— ·237287	— ·238835
0.6	— ·086238	— ·119983	— ·149462	— ·174951	— ·196669	— ·214788	— ·229442	— ·240740	— ·248755	— ·253542	— ·255134
0.7	— ·094122	— ·128626	— ·158725	— ·184720	— ·206847	— ·225295	— ·240206	— ·251695	— ·259847	— ·264714	— ·266333
0.8	— ·099285	— ·134240	— ·164705	— ·190999	— ·213369	— ·232010	— ·247074	— ·258679	— ·266911	— ·271826	— ·273459
0.9	— ·102115	— ·137299	— ·167953	— ·194401	— ·216897	— ·235637	— ·250781	— ·262444	— ·270719	— ·275655	— ·277299
a = 1.0	— ·102980	— ·138233	— ·168943	— ·195437	— ·217969	— ·236740	— ·251906	— ·263587	— ·271873	— ·276815	— ·278464

Table 15.

Imaginary part η of the roots $(\alpha, \beta, \gamma \ldots)$ of the equation $\zeta e^{-\zeta} = k \cdot a e^{-a}$; $k = \sqrt[x]{1}$. $x = 40$; $a = 0{\cdot}1.\ 0{\cdot}2,\ 0{\cdot}3 \ldots 0{\cdot}9$.

	0	1/40	2/40	3/40	4/40	5/40	6/40	7/40	8/40	9/40	10/40
$a = 0.0$	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
0.1	·000000	·017323	·033881	·048994	·062129	·072926	·081193	·086882	·090051	·090831	·089403
0.2	·000000	·038727	·074400	·104815	·128916	·146574	·158201	·164448	·166015	·163569	·157709
0.3	·000000	·065550	·121968	·165161	·195405	·214653	·225014	·228275	·225851	·218840	·208102
0.4	·000000	·099309	·175060	·225131	·255706	·272345	·278869	·277844	·271030	·259681	·244718
0.5	·000000	·140791	·229093	·278872	·305677	·317840	·319963	·314811	·304173	·289276	·271004
0.6	·000000	·187703	·277641	·322325	·343970	·351646	·349911	·341400	·327789	·310219	·289509
0.7	·000000	·232858	·315798	·354223	·371211	·375281	·370622	·359653	·343917	·324465	·302060
0.8	·000000	·267861	·341843	·375210	·388840	·390434	·383822	·371240	·354125	·333465	·309976
0.9	·000000	·288561	·356349	·386696	·398410	·398620	·390933	·377470	·359605	·338290	·314216
= 1.0	·000000	·295065	·360808	·390204	·401323	·401108	·393090	·379358	·361266	·339752	·315499

	10/40	11/40	12/40	13/40	14/40	15/40	16/40	17/40	18/40	19/40	20/40
$a = 0.0$	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
0.1	·089403	·085972	·080755	·073971	·065837	·056563	·046353	·035403	·023904	·012043	·000000
0.2	·157709	·148959	·137776	·124557	·109649	·093357	·075956	·057696	·038807	·019505	·000000
0.3	·208102	·194317	·178032	·159696	·139868	·118320	·095878	·072607	·048732	·024463	·000000
0.4	·244718	·226840	·206596	·184423	·160682	·135679	·109676	·082903	·055572	·027876	·000000
0.5	·271004	·250020	·226838	·201868	·175444	·147850	·119329	·090096	·060345	·030256	·000000
0.6	·289509	·266273	·240987	·214030	·185716	·156306	·126028	·095083	·063652	·031904	·000000
0.7	·302060	·277272	·250544	·222235	·192637	·161999	·130535	·098436	·065875	·033012	·000000
0.8	·309976	·284199	·256557	·227393	·196985	·165573	·133363	·100540	·067270	·033707	·000000
0.9	·314216	·287907	·259775	·230151	·199310	·167483	·134875	·101664	·068015	·034078	·000000
$a = 1.0$	·315499	·289029	·260748	·230986	·200013	·168061	·135332	·102004	·068240	·034190	·000000

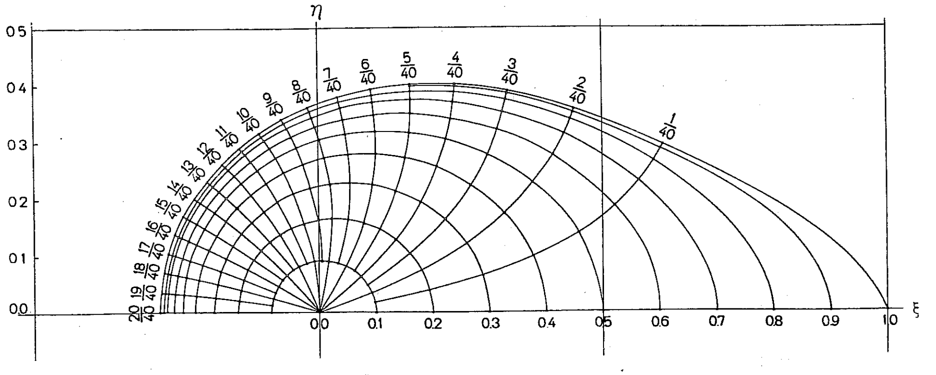
Table 16.

Mean waiting times (Two different hypotheses, T_1 and T_∞).

a \ x	1	2	4	5	8	10	20	40
0·1	0·11111	0·01010	0·00022	0·00004	0·00000	0·00000	0·00000	0·00000
	0·05556	0·00621	0·00016	0·00003	0·00000	0·00000	0·00000	0·00000
0·2	0·25000	0·04167	0·00299	0·00096	0·00004	0·00001	0·00000	0·00000
	0·12500	0·02423	0·00206	0·00069	0·00003	0·00001	0·00000	0·00000
0·3	0·42857	0·09890	0·01323	0·00575	0·00063	0·00017	0·00000	0·00000
	0·21429	0·05526	0·00846	0·00386	0·00047	0·00013	0·00000	0·00000
0·4	0·66667	0·1905	0·02779	0·01990	0·00385	0·00147	0·00002	0·00000
	0·33333	0·10331	0·02270	0·01243	0·00263	0·00105	0·00002	0·00000
0·5	1·00000	0·3333	0·08696	0·05215	0·01476	0·00722	0·00037	0·00000
	0·50000	0·17674	0·04965	0·03065	0·00932	0·00474	0·00028	0·00000
0·6	1·50000	0·5625	0·1794	0·1181	0·04361	0·02532	0·00302	0·00012
	0·75000	0·29304	0·09838	0·06607	0·02570	0·01535	0·00204	0·00009
0·7	2·33333	0·9608	0·3572	0·2519	0·1128	0·07391	0·01559	0·00181
	1·16667	0·49361	0·18971	0·13552	0·06268	0·04187	0·00954	0·00124
0·8	4·00000	1·7778	0·7456	0·5541	0.2860	0·2046	0·06402	0·01515
	2·00000	0·90328	0·38609	0·28906	0·15195	0·10983	0·03588	0·00907
0·9	9·00000	4·2632	1·9695	1·5250	0·8769	0·6687	0·2754	0·10288
	4·50000	2·14692	0·99997	0·77670	0·44997	0·34458	0·14429	0·05529

Table 17.

Roots ($\alpha, \beta, \gamma \ldots$) of the equation $\zeta e^{-\zeta} = k \cdot \alpha e^{-\alpha}$;
$k = \sqrt[x]{1}$; $\zeta = \xi + i\eta$; $x = 40$; $\alpha = 0{\cdot}1, 0{\cdot}2, \ldots 0{\cdot}9$.

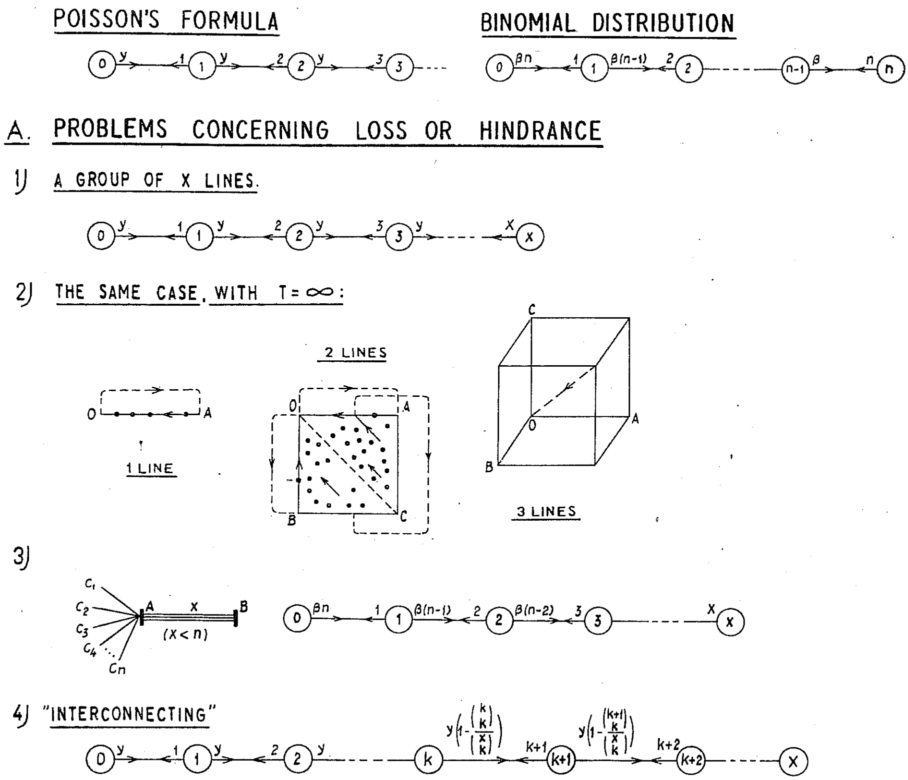


5. SOME APPLICATIONS OF THE METHOD OF STATISTIC EQUILIBRIUM IN THE THEORY OF PROBABILITIES

From the Sixth Scandinavian Mathematicians' Congress in Copenhagen, 1925; a lecture.

The problems of which I shall speak to-day belong to the Theory of Telephone Management, which is a branch, a comparatively new one, of the theory of probabilities. Accordingly we have to deal with telephone lines, telephone conversations, telephone exchanges, either manual or more or less automatic, and so on. The main thing is, however, the *calls* of the subscribers and their different fates. Some calls are "lucky" and obtain immediately the connexions desired, other calls are "unlucky" and either obtain no connexion or will have to wait a certain time. The reasons are various: the person called may be occupied with another telephone talk or otherwise; all the lines connecting the two exchanges are, momentarily, occupied by other conversations; or the telephone operators have not yet finished the work caused by preceding calls. In order to understand the real nature of the important practical questions arising in the manner indicated by these examples, one must undertake a theoretical work, based on the methods of probability; and this was shown, for the first time, about 20 years ago, by Mr. *F. Johannsen*, Managing Director of the Copenhagen Telephone Company. Within a short time he succeeded in throwing light upon several questions immediately urgent then and at all times important, and also in awaking a lively interest in this new field of research. Later on, owing to pressure of other work, he wished to leave the continuation of the work out of his hands, although not out of his eye-view and interest, and, therefore, trusted me with it; a task which has caused me much joy. I have reported the progress of the matter in several papers from 1909 onward, in Danish and other languages. In these I have given especially the exact resulting formulae of the most important and typical problems, also many tables with numerical results, but, for the sake of brevity, not always the complete demonstrations. See for instance a paper, printed in "Elektroteknikeren", 1923, and somewhat enlarged in "Annales des P. T. et T.", 1925; it was originally a lecture delivered at the H. C. Ørsted Centenary, 1920, and was also printed in the reports of this meeting, section electrotechnics,

Table 1.



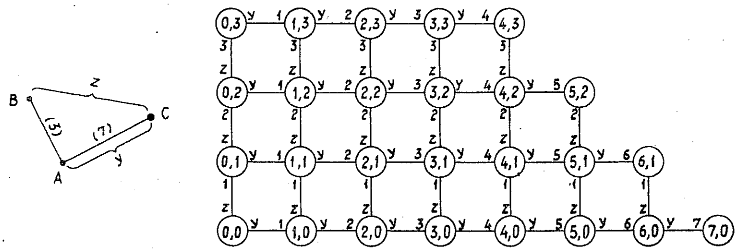
1922. In this paper may be found references to several contributions to the study of this class of problems, and to my own earlier papers. I will not state very many formulae, on this occasion, and not at all enter upon their numerical results; but I will lay stress on the method of reasoning, especially what may be called the method of statistical equilibrium, for obtaining distribution-laws.

1. We have to consider, at first, some very simple laws of distribution. We define the "traffic-intensity" as the product of the number of calls per unit of time into the holding time (the duration of the telephone conversation). It is, however, more convenient to take the unit of time equal to the duration of the conversation; the traffic-intensity is then simply equal to the number of calls per unit of time. We may suppose, for the present, that the duration is constant. If not, we have to speak of the mean duration instead of the duration. For the traffic-intensity we use the designation y.

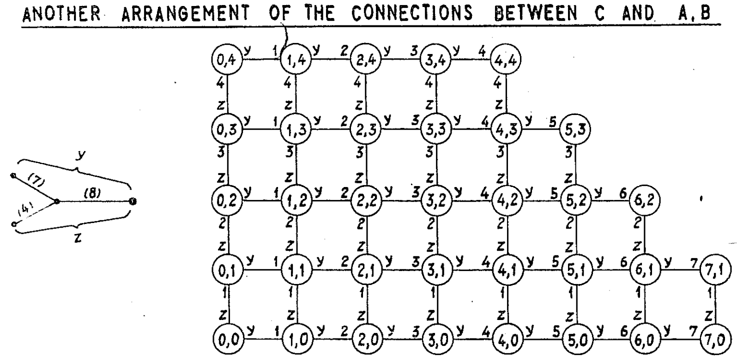
Now we have to consider 3 cases, differing in respect of the number and

Table 2.

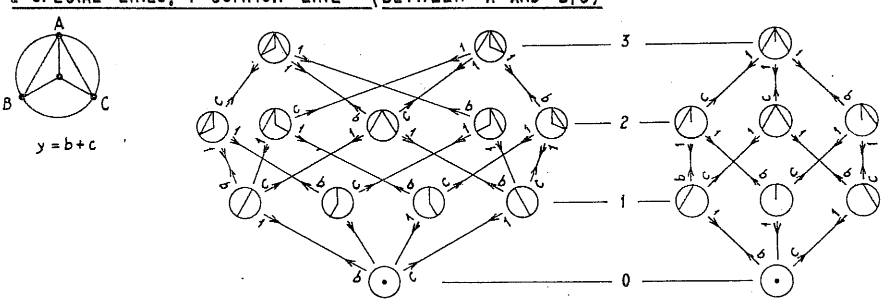
5) AN ARRANGEMENT OF THE CONNECTIONS BETWEEN C AND A,B



6) ANOTHER ARRANGEMENT OF THE CONNECTIONS BETWEEN C AND A,B



7) 2 SPECIAL LINES, 1 COMMON LINE (BETWEEN A AND B,C)



arrangement of the lines. In the first case we suppose that the number is so large that a want or shortage of lines will never occur, when the traffic-intensity is y. Now we put the question: What is the probability of 0, 1, 2... calls during a time equal to 1, or in a more tangible form, what is the probability of finding, at an arbitrary moment, 0, 1, 2... lines occupied by conversations? (It is obvious that the mean number of occupied lines is equal to y). For the probabilities of the different cases we may use the designations ((0)), ((1)), ((2))..., reserving the designations (0), (1), (2)... for the cases themselves. We can solve this problem, and likewise the following ones, in such a way that we find first the relative values of the probabilities and afterwards the absolute values (the sum of which is 1). We may use a graphical representation (see table 1); the different cases are here represented by a series of small circles, the transitions between the (successive) cases are indicated by connecting lines with arrows, and the probability-densities of the transitions per unit of time by the inscriptions. By means of a series of equations, which express the conditions of equilibrium or balance, we find easily the following relative values of the required probabilities, beginning with 1:

$$1;\; y;\; \frac{y^2}{2!};\; \frac{y^3}{3!} \ldots$$

If we divide by e^y, which is the sum of this series, we find the absolute values (see table 6), and we have now reached Poisson's law, which was originally, and is usually, derived in another way.

Among many remarkable properties of *Poisson's* function $\frac{e^{-y} y^x}{x!}$, I name one here (although it is not possible to enter upon its applications): If we consider the usual double-entry table of this function and take out all the values to be found along an oblique line, the slope of which is α, the sum of the values will be equal to $\frac{1}{1-\alpha}$, for $0 < \alpha < 1$ (and also for some other, not positive, values of α). This is a direct consequence of a theorem given in 1902, in Acta Mathematica, by the late Dr. *Jensen*, who through many years was the Chief Engineer of the Telephone Company, as we have just been reminded, and whose eminent position among Scandinavian Mathematicians is known to all present.

The next two problems are not quite so simple; they are in a certain way contrasts. The lines which go out from the telephone exchange, radiating in all directions, to the houses of the subscribers, are mutually independent. On the other hand, the lines or junctions which connect two exchanges, are cooperating; they are said to form a group or bundle. In the first of these two cases, a similar line of thought as the above will carry us into the binominal law of distribution, the law of *Pascal* and *Bernoulli*. Let n be the number of lines, β the traffic-intensity, or the number of calls per unit of time, per free line. We find the relative values of probabilities to be

$$1;\; \beta n;\; \beta^2 \frac{n(n-1)}{2!};\; \beta^3 \frac{n(n-1)(n-2)}{3!} \ldots$$

And then we obtain the absolute values by division by $(1 + \beta)^n$. In the specially simple case $n = 1$ we obtain $\frac{1}{1+\beta}$ and $\frac{\beta}{1+\beta}$; if instead of these quantities we put q and p, respectively, we may at once give the main result the perhaps more familiar form:

$$q^n;\ q^{n-1} \cdot p \cdot n;\ q^{n-2}\, p^2 \frac{n\,(n-1)}{2!};\ q^{n-3}\, p^3 \frac{n\,(n-1)\,(n-2)}{3!} \ldots.$$

(It is easy to show that these results do not necessarily imply the hypothesis of constant duration).

Now let the lines go side by side and form a group, within which each line may replace each other; a case more important than the preceding. Let x be the number of lines. In this case the inflow of fresh conversations will not decrease gradually in proportion as the lines are occupied, but the moment all lines are busy, and only then, the inflow will suddenly stop. We find for the required probabilities (again by use of the equilibrium-principle) firstly the relative values

$$1;\ y;\ \frac{y^2}{2!};\ \frac{y^3}{3!} \ldots \frac{y^x}{x!};$$

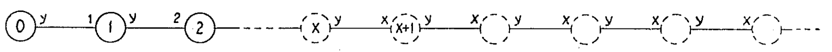
and, hence, secondly, the absolute values (see the collection of formulae, table 6). As the probability of finding all lines occupied is the same thing as the proportion B of "unlucky" or "lost" calls to all calls, sometimes called the degree of hindrance or obstruction, it is much more important in practice than the other ones. — The product $B \cdot y$ is the number of lost calls per unit of time. If the difference of this quantity with respect to x is fixed to be about a certain constant (depending on the price of a line per unit of time and on the value lost with the call), then we may say that x has been correctly determined, as a function of y. But we must leave the further consideration of this theme.

It will now be necessary to insert here some remarks about the law of distribution (or frequency) of the durations. For although in many cases, for instance in the case just considered, the results are really independent of the nature of the distribution, it does not follow that the demonstration can be given with the same words, or as easily, for one law as for another. The laws which mostly deserve our attention may be seen in the collection of formulae table 6; they constitute an infinite series, and the successive terms are marked $T = 1,\ T = 2 \ldots T = \infty$. As stated here, the formulae give the probability of a duration greater than n (the mean duration is always equal to 1); by a differentiation one may obtain the density of probability, which is another expression of

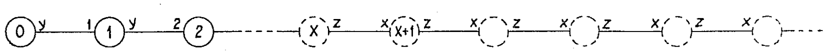
Table 3.

B. WAITING TIMES

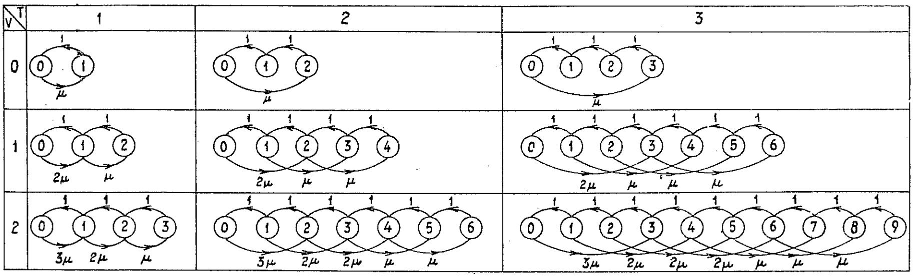
1) THE MAIN CASE : A GROUP OF X LINES.



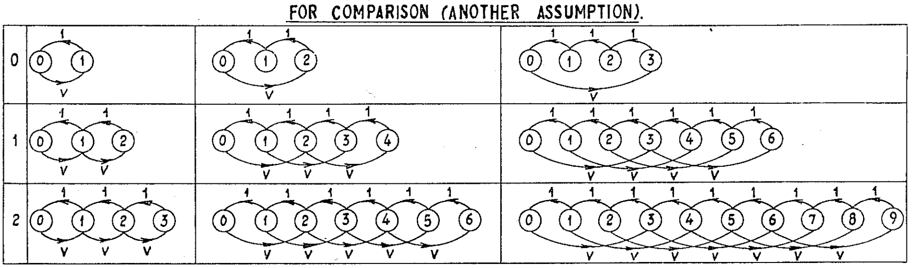
2) A MODIFICATION. z < y



3) DIFF. SYSTEMS OF DISTRIBUTION, HALF POSITIONS (PARTIAL POSITIONS), BIG GROUPS

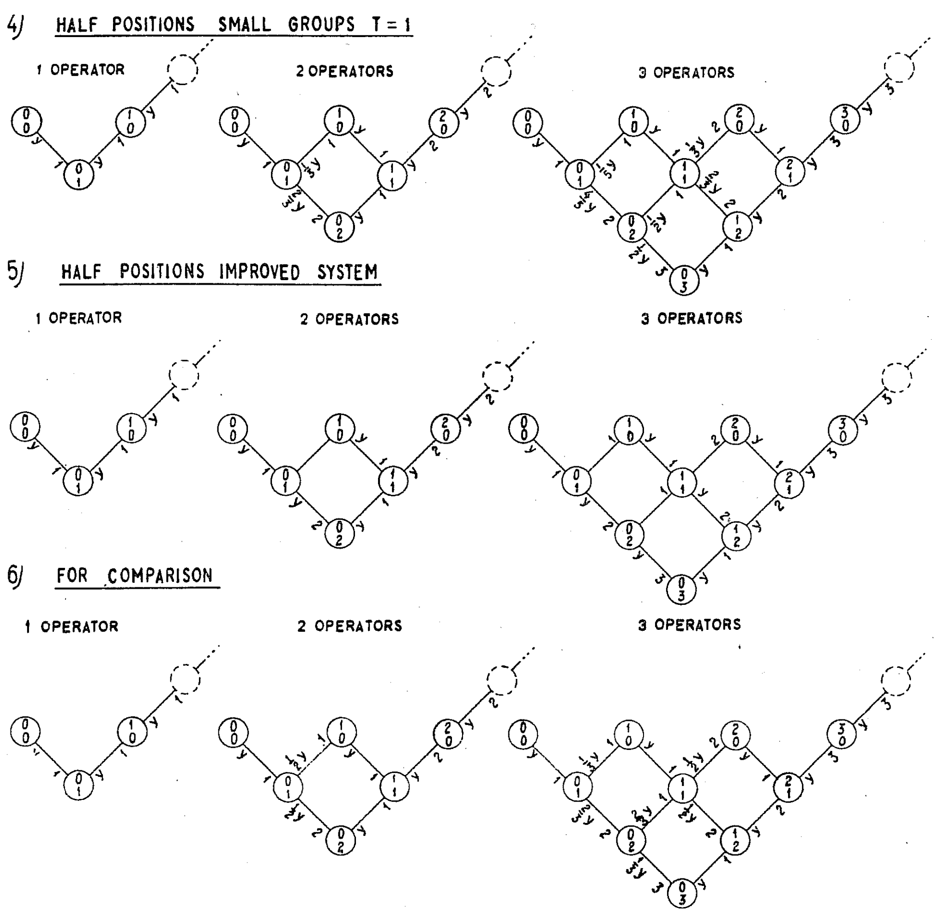


FOR COMPARISON (ANOTHER ASSUMPTION).



the law. Compare table 5. The simplest and most important law is, no doubt, the first one; we come to this law if we suppose that the "force of mortality" (we borrow this word from another domain) is independent of the "age". About 200 years ago it was commonly believed, in Denmark at least, that the probability of death within a year was the same for all ages. We know that this is wrong; although no doubt some causes of death will hit the young and the old indifferently. But the life of a telephone conversation comports very well with the simple rule, as borne out by practical tests. Only some very special classes of conversations, for instance those caused by the communication of the "number" to the telephone operator, must be excepted, as they will not agree with this law, but more closely with one of the following ones. For the rest all these laws are closely related. For instance, No. 2 may be taken as the law of

Table 4.



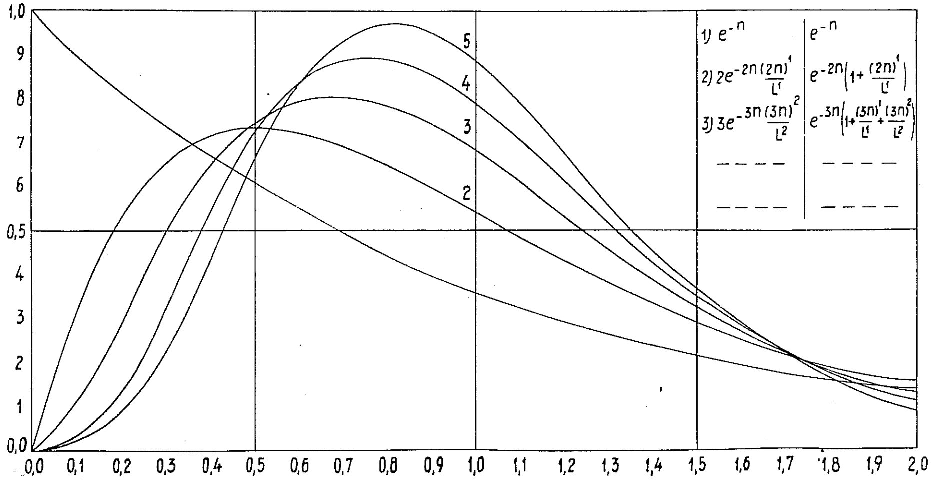
the duration of a compound conversation, if the two components are of the type No. 1, with a mean duration of $\frac{1}{2}$, and so on. Conversely, the "compound" conversations may be dissected into simple ones, and in this manner we may simplify many problems. The expressions of the probability of a duration exceeding n may now be written down immediately (by means of Poisson's formula). The limiting case, $T = \infty$, is obviously identical with the case of constant duration.

Now we return to the demonstration postponed and start with the assumption $T = 1$; here it is possible to treat each of the cases (0), (1), (2)... (x) (that is to say 0, 1, 2... x lines occupied) without regarding the respective ages of the conversations, and accordingly the demonstration is straightforward. On the other hand, if $T = \infty$, the ages, or better the

Table 5.

DURATION OF A TELEPHONE CONVERSATION:

DIFFERENT POSSIBLE LAWS OF DISTRIBUTION.



residual lifetimes, are important and must be specified; for this purpose we may use, for $x = 1$, 2, and 3, respectively, a line-segment, a square, and a cube. For $x = 2$, for instance, we may place a point within the square, or on the sides OA and OB, or in the corner O. If, now, we use an infinite number of points, if, further, we strew evenly all parts of the square, and also evenly all points of the sides OA and OB, not forgetting the corner point O, and if we choose properly the relative probabilities of the three main cases, then (and only then) we can easily show the existence of a statistical equilibrium. This equilibrium is caused by all the changes forthcoming as a consequence either of the progress of time or of the arrival of new calls. — If $x > 3$, we must do without a figure, but otherwise we can reason in the same manner.

I should like to give also a proof comprehending at once all laws of distribution; I will only say that it is suggested by a comparison of the two cases; firstly an infinite number of lines, secondly, x lines; the relative values of the x probabilities ((0)), ((1)), ((2))... ((x)) are, and must be, the same in both cases.

2. We will consider a few different line-systems, most of them a little more complicated than the preceding ones. In the figure 3 (table 1) an important case is represented; it is, in a way, a combination of the two just treated. We see, to the right, x lines running side by side, between the two points (exchanges) A and B, and to the left we see n lines (sub-

scribers' lines) radiating in all directions to the points C_1, $C_2 \dots C_n$. The calls may originate from the left or from the right, indifferently. The traffic-intensity, or the number of calls per unit of time, per subscriber (unoccupied) is β. We can have here, at most, x conversations running at the same time. We easily find the fundamental probabilities ((0)), ((1)), ((2))... ((x)), and afterwards other probabilities, the most important being the probability of the existence, or non-existence, at an arbitrary moment, of a free way *C-A-B* between a certain subscriber *C* and the exchange *B*.

In fig. 4 we have a system of x lines, which constitute a group. Each forthcoming call will try (automatically) a number k of the lines (not all the lines); if the k lines are engaged, the call is lost. For the rest, the choice of the k lines is a new one for each subsequent call, and upon the whole, they are selected in as many ways as possible. The figure indicates the beginning of the solution of the problem; the resulting formulae and also numerical tables have been given elsewhere. This system belongs to a type ordinarily named "systems with 'interconnecting'".

In fig. 5 we have a group of 3 lines between *A* and *B*, 7 between *B* and *C*; *C* is the main telephone-exchange, to which or from which (almost) all the traffic is going; the traffic y between *B* and *C*, and z between *A* and *C*. The different possible cases are represented in the oblique quadrangle as circles; the cases represented at the top of this figure and to the right are especially important. The conditions of equilibrium may be written down by considering the "transitions" between each case (circle) and its surroundings in the figure. The same relative values may be derived in a simpler way, if we consider the horizontal or vertical connecting lines, one at a time; for each single line we find (in this problem) a special balance.

The problem of fig. 6 is of a similar type.

The problem of fig. 7 is one resolved by Dr. *A. E. Vaulot* (in Revue Génerale de l'Electricité, 1924); we have here 2 special lines, *AB* and *AC*, and a common line acting as a reserve for both.

3. Let us consider a group of x lines, with a traffic intensity y. Let us suppose that a call, which cannot obtain a line at once, will keep waiting and thereby obtain a line, in its turn. For simplicity, we consider here only the hypothesis $T = 1$. The series of possible cases is not limited to (0), (1)... (x), but it goes on with ($x + 1$), ($x + 2$)..., that is, in addition to the calls which have already found a connexion, we may have 1, 2... calls waiting. See table 3. — We easily find the required probabilities ((0)), ((1))... ((x)), (($x + 1$)), (($x + 2$))...; the last ones, from ((x)) onward, are the most interesting. In table 7 are found the expressions of these by means of their sum c, and further the expression of c by $D(x)$ and $D(x - 1)$, defined at the same place as functions of x and y.

Table 6.

POISSON'S FORMULA
(A large number of lines).

$$\begin{cases} S_0 = e^{-y} \\ S_1 = e^{-y} \cdot y \\ S_2 = e^{-y} \cdot \dfrac{y^2}{2!} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \end{cases}$$

BINOMIAL DISTRIBUTION
(n independent lines).

$$\begin{cases} S_0 = 1 \cdot \dfrac{1}{(1+\beta)^n} \\ S_1 = \dfrac{\beta n}{1} \cdot \dfrac{1}{(1+\beta)^n} \\ S_2 = \dfrac{\beta^2 \cdot n(n-1)}{2!} \cdot \dfrac{1}{(1+\beta)^n} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ S_n = \beta^n \cdot \dfrac{1}{(1+\beta)^n} \end{cases}$$

LOSS OR HINDRANCE
(A group of x lines).

$$\begin{cases} S_0 = \dfrac{1}{1 + y + \dfrac{y^2}{2!} + \dots \dfrac{y^x}{x!}} \\ S_1 = \dfrac{y}{1 + y + \dfrac{y^2}{2!} + \dots \dfrac{y^x}{x!}} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ S_x = \dfrac{\dfrac{y^x}{x!}}{1 + y + \dfrac{y^2}{2!} + \dots \dfrac{y^x}{x!}} \end{cases}$$

DIFFERENT LAWS OF DISTRIBUTION FOR THE DURATION

$(T = 1)$	$S = e^{-n}$
$(T = 2)$	$S = e^{-2n}\left(1 + \dfrac{2n}{1!}\right)$
$(T = 3)$	$S = e^{-3n}\left(1 + \dfrac{3n}{1!} + \dfrac{(3n)^2}{2!}\right)$
.	. .
.	. .
.	. .
.	. .

Table 7.

WAITING-TIMES
(A group of x lines, $T = 1$).

$$\left\{\begin{aligned} ((x)) &= c \cdot \frac{x-y}{x} \\ ((x+1)) &= ((x)) \cdot \frac{y}{x} \\ ((x+2)) &= ((x+1)) \cdot \frac{y}{x} \\ &\vdots \end{aligned}\right.$$

$$\left\{\begin{aligned} c &= \frac{1}{D(x) - D(x-1)} \\ D(x) &= \frac{1 + y + \frac{y^2}{2!} + \ldots \frac{y^x}{x!}}{\frac{y^x}{x!}} \\ D(x-1) &= \frac{1 + y + \frac{y^2}{2!} + \ldots \frac{y^{x-1}}{(x-1)!}}{\frac{y^{x-1}}{(x-1)!}} \end{aligned}\right.$$

$$\left\{\begin{aligned} S(>n) &= ((x)) \cdot e^{-xn} + ((x+1)) \cdot e^{-xn}\left[1 + \frac{xn}{1!}\right] + \\ &\quad ((x+2)) \cdot e^{-xn}\left[1 + \frac{xn}{1!} + \frac{(xn)^2}{2!}\right] + \ldots \\ S(>n) &= e^{-xn}\left[((x)) + ((x+1)) + \ldots\right] + \\ &\quad e^{-xn} \cdot \frac{xn}{1!}\left[((x+1)) + (x+2)) + \ldots\right] + \\ &\quad e^{-xn} \cdot \frac{(xn)^2}{2!}\left[((x+2)) + ((x+3)) + \ldots\right] + \ldots \end{aligned}\right.$$

$$S(>n) = c \cdot e^{-(x-y)n}$$

$$\left\{\begin{aligned} M &= \left[1 \cdot ((x+1)) + 2 \cdot ((x+2)) + 3 \cdot ((x+3)) + \ldots\right] \frac{1}{y} \\ M &= \left[1 \cdot ((x)) + 2 \cdot ((x+1)) + 3 \cdot ((x+2)) + \ldots\right] \frac{1}{x} \end{aligned}\right.$$

$$M = c \cdot \frac{1}{x-y}$$

Now we are able to find also the probability, $S\,(>n)$, of a waiting-time exceeding n. If we take a view of the chances of this eventuality, we see at once that these depend partly on the number of calls waiting already, partly on the number of conversations coming to an end in the course of the time n. We find the probability in question in the shape of a doubly infinite series and by summation we get the result:

$$S\,(>n) = c \cdot e^{-(x-y)\,n}.$$

Further, it is very desirable to find the mean waiting-time M; one can perform this in different ways, f. inst. by means of the formula just mentioned, or else more directly by means of one of the two equations (each of them easily found):

$$M = \left[1 \cdot ((x+1)) + 2 \cdot ((x+2)) + 3 \cdot ((x+3)) \ldots\right] \cdot \frac{1}{y}$$

$$M = \left[1 \cdot ((x)) + 2 \cdot ((x+1)) + 3 \cdot ((x+2)) \ldots\right] \cdot \frac{1}{x}$$

In each of these ways we find:

$$M = \frac{c}{x-y}.$$

We have supposed, that the lines when disengaged are given to the waiting calls in turn. If, however, they are allotted purely at random which is often the case in practice, we have to deal with another (and more difficult) problem; mark that both $S\,(>0)$ and M retain the same values as formerly.

Another modification of the problem is met with, if a certain fractional part of the callers will not wait, and the rest will wait (see fig. 2). It is easy to find out the consequences of this, for the one and the other part.

4. All the just mentioned results concerning waiting-times are valid not only for a group of x lines, but also for a group of x operators, whose common task it is to set up the connexions required by a number of subscribers. As is well known, it is possible to build up a telephone-system for a large town without operators. But the technical investigations undertaken especially by Mr. *P. V. Christensen*, Chief Engineer of our Company, have shown that the economy is doubtful at least in the case of Copenhagen; also by the "automatic" system a great amount of work is laid upon the subscribers themselves. For the present, and probably in the future, the operators will be retained in Copenhagen, but automatic devices are provided for the distribution of the calls, in order to direct each call to an operator who either is ready or, at least, probably will be ready very soon. The systems used in the most modern parts of our main telephone-exchange and also in some other exchanges, are called

Table 8.

AUTOMATIC DISTRIBUTION, HALF-POSITIONS; BIG GROUPS
Supposition: Half chance, if one half-position is unoccupied.

	$T=1$	$T=2$	$T=3$
((0))	$((0)) = 1 \cdot a$	$((0)) = 1 \cdot a$	$((0)) = 1 \cdot a$
((1))	$((1)) = 2\mu \cdot a$	$((1)) = 2\mu \cdot a$	$((1)) = 2\mu \cdot a$
((2))	$((2)) = 2\mu^2 \cdot a$	$((2)) = (2\mu+2\mu^2) \cdot a$	$((2)) = (2\mu+2\mu^2) \cdot a$
((3))		$((3)) = (4\mu^2+2\mu^3) \cdot a$	$((3)) = (2\mu+4\mu^2+2\mu^3) \cdot a$
((4))		$((4)) = (2\mu^2+2\mu^3) \cdot a$	$((4)) = (6\mu^2+6\mu^3+2\mu^4) \cdot a$
((5))			$((5)) = (4\mu^2+6\mu^3+2\mu^4) \cdot a$
((6))			$((6)) = (2\mu^2+4\mu^3+2\mu^4) \cdot a$
$\frac{1}{a} =$	$1+2\mu+2\mu^2 =$ $1+2\mu\,(1+\mu)$	$1+4\mu+8\mu^2+4\mu^3 =$ $1+4\mu\,(1+\mu)^2$	$1+6\mu+18\mu^2+18\mu^3+6\mu^4 =$ $1+6\mu\,(1+\mu)^3$

Supposition: Full chance, if one half-position is unoccupied.
(For comparison with the above).

	$T=1$	$T=2$	$T=3$
((0))	$((0)) = 1 \cdot a$	$((0)) = 1 \cdot a$	$((0)) = 1 \cdot a$
((1))	$((1)) = \nu \cdot a$	$((1)) = \nu \cdot a$	$((1)) = \nu \cdot a$
((2))	$((2)) = \nu^2 \cdot a$	$((2)) = (\nu+\nu^2) \cdot a$	$((2)) = (\nu+\nu^2) \cdot a$
((3))		$((3)) = (2\nu^2+\nu^3) \cdot a$	$((3)) = (\nu+2\nu^2+\nu^3) \cdot a$
((4))		$((4)) = (\nu^2+\nu^3) \cdot a$	$((4)) = (3\nu^2+3\nu^3+\nu^4) \cdot a$
((5))			$((5)) = (2\nu^2+3\nu^3+\nu^4) \cdot a$
((6))			$((6)) = (\nu^2+2\nu^3+\nu^4) \cdot a$
$\frac{1}{a} =$	$1+\nu+\nu^2 =$ $1+\nu(1+\nu)$	$1+2\nu+4\nu^2+2\nu^3 =$ $1+2\nu\,(1+\nu)^2$	$1+3\nu+9\nu^2+9\nu^3+3\nu^4 =$ $1+3\nu\,(1+\nu)^3$

Table 9.

AUTOMATIC DISTRIBUTION, HALF-POSITIONS, BIG GROUPS

Supposition: Half chance, if one half-position is unoccupied.

	$T = 1$	$T = 2$	$T = 3$	$T = \infty$
$\alpha =$	$\frac{2\mu(1+\mu)}{1+2\mu(1+\mu)}$	$\frac{4\mu(1+\mu)^2}{1+4\mu(1+\mu)^2}$	$\frac{6\mu(1+\mu)^3}{1+6\mu(1+\mu)^3}$	$\frac{2g}{2g+e^{-g}}$ ($g = \lim T\mu$)
$M =$	$\frac{\mu}{1+\mu}$	$\frac{\mu(1+2(1+\mu))}{2(1+\mu)^2}$	$\frac{\mu(1+2(1+\mu)+3(1+\mu)^2)}{3(1+\mu)^3}$	$\frac{g-1+e^{-g}}{g}$
$S(>) =$	$\mu\frac{1}{1+\mu}\cdot e^{-n}$	$\mu\left(\frac{1}{(1+\mu)^2}+\frac{1+2n}{1+\mu}\right)e^{-2n}$	$\mu\left(\frac{1}{(1+\mu)^3}+\frac{1+3n}{(1+\mu)^2}+\frac{1+3n+\frac{9n^2}{2}}{1+\mu}\right)e^{-3n}$	$1-e^{-g(1-n)}$ ($n < 1$)

Supposition: Full chance, if one half-position is unoccupied. (For comparison with the above).

	$T = 1$	$T = 2$	$T = 3$	$T = \infty$
$\alpha =$	$\frac{\nu(1+\nu)}{1+\nu(1+\nu)}$	$\frac{2\nu(1+\nu)^2}{1+2\nu(1+\nu)^2}$	$\frac{3\nu(1+\nu)^3}{1+3\nu(1+\nu)^3}$	$\frac{2h}{2h+e^{-h}}$ ($h = \lim T\nu$)
$M =$	$\frac{\nu}{1+\nu}$	$\frac{\nu(1+2(1+\nu))}{2(1+\nu)^2}$	$\frac{\nu(1+2(1+\nu)+3(1+\nu)^2)}{3(1+\nu)^3}$	$\frac{h-1+e^{-h}}{h}$
$S(>) =$	$\nu\frac{1}{1+\nu}\cdot e^{-n}$	$\nu\left(\frac{1}{(1+\nu)^2}+\frac{1+2n}{1+\nu}\right)e^{-2n}$	$\nu\left(\frac{1}{(1+\nu)^3}+\frac{1+3n}{(1+\nu)^2}+\frac{1+3n+\frac{9n^2}{2}}{1+\nu}\right)e^{-3n}$	$1-e^{-h(1-n)}$ ($n < 1$)

"half-position systems". Each operator has two "half-positions"; the half-position which she is serving at the moment is barred for access, the other one is free, if no call is waiting there, and barred if one call is waiting. (Instead of giving the operator two half-positions it is, of course, also possible to give her three third-part-positions, and so on). If we will study this system closely, we notice that two modifications present themselves naturally for choice: the operator, who has one half-position unoccupied, may have just the same chance as the operator who has two, or she may have only half a chance; the last arrangement is generally preferred in practice. We may very often content ourselves with studying the case of a large group of operators, which is relatively simple. We may further suppose $T = 1$, or 2, or 3.... The different stages or degrees of occupation of an operator are represented in fig. 3, table 3, by the circles. In table 8 is further shown how to find the corresponding fundamental probabilities; and the formulae hence resulting, of $S\,(>)$ and of M, are shown in table 9.

For small groups of operators, f. inst. for $x = 1, 2, 3 \ldots$, the problem is somewhat more difficult, and therefore I have treated it only with the supposition $T = 1$ (and only the half-positions, not the third-part positions, &c.). See figs. 4, 5, 6, table 4. In one of these figures, fig. 5, is shown a new and improved form of the half-position system. In this system an operator, who has only one half-position free, has absolutely no chance of being chosen if, at the same time, it is possible to find an operator who has two. The manner of action of this system will be clearly indicated by the figure.

If time had permitted, it would have been interesting to draw some connecting lines into other known applications of the conception of statistical equilibrium. I can just name: 1) *Euler*'s theory on the age-distribution of a population, about 1750, see Opera, series 1, vol. 7; a special case already treated by *Halley*; new contributions by *Lotka* and by *Nybølle*; 2) *Maxwell's* famous law of distribution for the velocities of gas molecules (an elementary proof, by myself, in "Fysisk Tidsskrift" 1925); and 3) the very interesting researches on the ultimate consequences of *Mendel's* laws of heredity, by *Hardy*, *Hagström*, *Bernstein*.

6. ON THE RATIONAL DETERMINATION OF THE NUMBER OF CIRCUITS

Written in 1924. First published in the present book.

1. Systems With Barred Access.

One might, perhaps, think that, for the determination of the number of circuits required, x, for a certain amount of traffic, y, or *vice versa*, it would be necessary to know only the formula of the probability of barred access:

$$B = \frac{\frac{y^x}{x!}}{1 + y + \frac{y^2}{2!} + \ldots . \frac{y^x}{x!}}, \tag{1}$$

combined with accordance in opinion as to the fixing of a suitable value of B; in that case, a table of the co-ordinate values of x and y would be sufficient. There are, however, the following points to be considered: in the first place, B must be multiplied by y (y designating the number of calls being originated during the unit of time, *i. e.* the average length of conversation), whereby the number of calls being barred during the unit of time is obtained; furthermore, it is not exactly $B \cdot y$ we want, but the reduction of $B \cdot y$ that will take place the moment we increase the number of available circuits by 1; this reduction we may call the "*Improvement*," F_1 (in Danish: "*Forbedringen*"). When $B\ (x - 1)$ and $B\ (x)$ denote the values of B corresponding to $x - 1$ and x, the improvement caused by the transition from $x - 1$ circuits to x will be, then,

$$F_1 = y \cdot B\ (x - 1) - y \cdot B\ (x) \tag{2}$$

Table 1 gives F_1 for different values of x and y. (There are special reasons — which will be obvious later — for the method according to which the values of y have been selected). It will be noticed that F_1 decreases gradually as x increases. In order to find the particular value of x that is neither too great nor too small (for a certain value of y), it will be necessary to use a certain value of F_1 as our basis, and seek out in the table the place where one of two consecutive values of F_1 is slightly greater, and the other slightly smaller, than the basis value of F_1. In determining this basis value of F_1, allowance must be made for the inconvenience caused by preventing a call from getting through (this may be expressed in terms of money); also, for the costs per unit of time being incurred owing to the new circuit; the said costs comprise interest on the invested capital, depreciation, and maintenance of the circuit with accessories). If the employment of this, the most direct, method should give rise to any difficulties, it is of course possible to use another method; a single well-matched set of values of x and y could be taken as starting point; the values of F_1 for the said values of x and y could be found in the table; and the rest would pass off as described above. In all essentials, this chain of reasoning applies to the circuits interconnecting two exchanges as well as to the lines between the exchange and a subscriber with several lines.

Note: Instead of the above mentioned (exact) procedure, another (approximative) method may be used; the latter is especially useful and convenient for the greater values of x and y. The approximation improving with increasing values of x, we get:

$$F_1 = \frac{h \cdot \phi}{\phi_{-1}} + \frac{\phi^2}{\phi_{-1}^2}, \tag{3}$$

where

$$x = y + h\sqrt{y} \tag{4}$$

(here, ϕ and ϕ_{-1} denote simple and wellknown functions of h, *viz.* the Gaussian law-of-error function and its integral). The above shall not be proved here; but its correctness will appear rather clearly from the main table, 1, by traversing the latter along oblique lines issuing from the left-hand top corner. In consequence of the manner in which the values of y as contained in the table have been chosen, we have all along each line $x = y + h\sqrt{y}$, where h is constant. The values of F_1 are approximately constant along such a line, especially for the greater values of x, and consistent with formula (3).

A formula of the type

$$x = y + h\sqrt{y}$$

has already been employed for a long time in our Company to determine the number of junctions[1]), although without any absolutely satisfactory statement of reasons, as far as I know. The reasons can now be explained by means of the foregoing, also the proper significance of the constant h.

[1]) This formula was published by *P. V. Christensen* in 1913 (see the foot-note p. 16).

Table 1. The improvement F_1. (Formula 2).

$x-y$	$y=1$	$y=4$	$y=9$	$y=16$	$y=25$	$y=36$	$y=49$	$y=64$	$y=81$	$y=100$
0	0.5000	0.5601	0.5837	0.5963	0.6040	0.6092	0.6131	0.6159	0.6181	0.6200
1	0.3000	0.4464	0.5072	0.5389	0.5582	0.5714	0.5807	0.5878	0.5933	0.5975
2	0.1375	0.3276	0.4242	0.4770	0.5094	0.5311	0.5465	0.5580	0.5671	0.5742
3	0.0471	0.2177	0.3397	0.4121	0.4577	0.4884	0.5104	0.5269	0.5394	0.5495
4	0.0123	0.1293	0.2581	0.3463	0.4045	0.4444	0.4731	0.4945	0.5111	0.5244
5	0.0026	0.0683	0.1856	0.2821	0.3506	0.3991	0.4345	0.4612	0.4819	0.4983
6	0.0004	0.0321	0.1252	0.2221	0.2977	0.3537	0.3953	0.4270	0.4517	0.4713
7	0.0001	0.0135	0.0793	0.1684	0.2471	0.3086	0.3558	0.3923	0.4210	0.4441
8		0.0051	0.0471	0.1229	0.2001	0.2650	0.3167	0.3575	0.3900	0.4162
9		0.0018	0.0263	0.0861	0.1579	0.2237	0.2785	0.3229	0.3588	0.3881
10		0.0006	0.0138	0.0579	0.1213	0.1854	0.2419	0.2891	0.3279	0.3599
11		0.0002	0.0067	0.0374	0.0906	0.1507	0.2071	0.2561	0.2974	0.3320
12		0.0001	0.0032	0.0231	0.0658	0.1201	0.1749	0.2246	0.2676	0.3043
13			0.0014	0.0138	0.0464	0.0938	0.1456	0.1948	0.2388	0.2769
14			0.0006	0.0079	0.0318	0.0718	0.1192	0.1671	0.2113	0.2506
15			0.0003	0.0044	0.0212	0.0538	0.0963	0.1417	0.1853	0.2251
16			0.0001	0.0023	0.0138	0.0394	0.0765	0.1186	0.1610	0.2008
17				0.0012	0.0087	0.0283	0.0698	0.0982	0.1386	0.1778
18				0.0006	0.0054	0.0199	0.0460	0.0802	0.1181	0.1561
19				0.0003	0.0032	0.0138	0.0348	0.0647	0.0997	0.1361
20				0.0001	0.0019	0.0093	0.0260	0.0616	0.0833	0.1177
21				0.0001	0.0011	0.0062	0.0190	0.0406	0.0689	0.1009
22					0.0006	0.0040	0.0137	0.0315	0.0564	0.0859
23					0.0003	0.0026	0.0097	0.0242	0.0457	0.0724
24					0.0002	0.0016	0.0068	0.0183	0.0367	0.0606
25					0.0001	0.0010	0.0048	0.0137	0.0291	0.0502
26						0.0006	0.0032	0.0101	0.0229	0.0413
27						0.0004	0.0021	0.0074	0.0178	0.0337
28						0.0002	0.0014	0.0053	0.0137	0.0272
29						0.0001	0.0009	0.0038	0.0104	0.0218
30						0.0001	0.0006	0.0027	0.0079	0.0173
31							0.0004	0.0019	0.0059	0.0136
32							0.0002	0.0013	0.0043	0.0107
33							0.0001	0.0009	0.0032	0.0083
34							0.0001	0.0006	0.0023	0.0063
35							0.0001	0.0004	0.0017	0.0048
36								0.0003	0.0012	0.0037
37								0.0002	0.0008	0.0027
38								0.0001	0.0006	0.0020
39								0.0001	0.0004	0.0015
40									0.0003	0.0011
41									0.0002	0.0008
42									0.0001	0.0006
43									0.0001	0.0004
44									0.0001	0.0003
45										0.0002

2. Systems With Delay.

Here, as in the preceding section, we must find out about the improvement obtained by adding 1 circuit to those already existing.

We may start from the average delay formula for x lines, supposing exponentially distributed holding time:

$$M(x) = \frac{1}{x - y} \cdot \frac{1}{D(x) - D(x-1)}, \tag{5}$$

where

$$D(x) = \frac{1}{B(x)} \quad \text{and} \quad D(x-1) = \frac{1}{B(x-1)}$$

and $B(x)$ and $B(x-1)$ have the same significance as above. Since, on the average, y calls are originated during the unit of time, we have to multiply by y, and so we get:

$$y \cdot M(x) = \frac{y}{x - y} \cdot \frac{1}{D(x) - D(x-1)},$$

which is the total average delay per unit of time, or, if you like, the mean number of simultaneously waiting calls. For $x - 1$ we have, correspondingly,

$$y \cdot M(x-1) = \frac{y}{x - 1 - y} \cdot \frac{1}{D(x-1) - D(x-2)}.$$

Hence, by transition from $x - 1$ to x circuits the improvement will be:

$$F_2 = y \cdot M(x-1) - y \cdot M(x) = \frac{y}{x-1-y} \cdot \frac{1}{D(x-1) - D(x-2)} - \frac{y}{x-y} \cdot \frac{1}{D(x) - D(x-1)}. \tag{6}$$

Table 2 is a table of the quantities F_2 thus determined. As was the case in the above, F_2 will be found to decrease as x increases; consequentially, there must be an optimum point where F_2 will about balance against the costs in connexion with adding one more circuit, or, expressed more exactly, the next improvement will be just too small to justify the costs of another new circuit. Although the word *circuit* has been used all along, it should be remembered that the delays may as well be due to the simultaneous engagement of each of a group of teamworking *operators*, instead of a group of cooperating circuits; and this leads to the question, How many operators should be set to work the traffic concerned? Mr. *K. Moe* has been working on this problem; as might be expected, his reasoning and procedure — although somewhat different in form — in reality agree with the above.

The results as stated in table 2 may be interpreted very clearly and plainly if we, in the following, rather stick to the last mentioned application of the theory. The whole problem consists of two points: the subscriber's waiting time, receiver in hand; and the operators' hours of service at the exchange (not exactly the time they are actually working their positions). It is necessary to have a certain basis value of F_2 in order to take full advantage of the table; now, it is obvious that if we put $F_2 = 1$, it means that an average subscriber's waiting time at his telephone is rated at the same value as the operator's hours of service at the exchange; $F_2 = \frac{1}{2}$ means that the subscriber's time is twice as valuable, &c.

Note: Here, too, another (approximative) method may be used instead of the (exact) procedure just mentioned, the former being especially useful and convenient for great values of x:

$$F_2 = \frac{\phi^2 (1 + h^2) + \phi_{-1} \phi (2h + h^3)}{(\phi + h\phi_{-1})^2 \cdot h^2}, \qquad (7)$$

$$x = y + h\sqrt{y}. \qquad (8)$$

The proof of this shall not be given here; but its correctness will appear rather clearly from the main table, 2, by transversing the latter along oblique lines issuing from the left-hand top corner. In consequence of the manner, in which the values of y as contained in the table have been chosen, we have $x = y + h\sqrt{y}$ all along such a line. It will be noticed that the values of F_2 are approximately constant along such a line, especially for the greater values of x, and consistent with formula (7).

Table 2. The improvement F_2. (Formula 6).

$x-y$	$y=1$	$y=4$	$y=9$	$y=16$	$y=25$	$y=36$	$y=49$	$y=64$	$y=81$	$y=100$
2	0.288	1.647	4.081	7.545	12.025	17.505	23.995	31.487	39.961	49.465
3	0.039	0.389	1.139	2.257	3.726	5.537	7.689	10.181	13.007	16.168
4	0.006	0.121	0.443	0.968	1.680	2.570	3.634	4.864	6.272	7.845
5	0.001	0.040	0.194	0.477	0.907	1.400	2.026	2.760	3.596	4.532
6		0.013	0.088	0.251	0.500	0.830	1.236	1.715	2.263	2.884
7		0.004	0.040	0.136	0.296	0.517	0.795	1.128	1.514	1.951
8		0.001	0.018	0.074	0.176	0.331	0.530	0.771	1.053	1.374
9			0.008	0.040	0.109	0.216	0.360	0.539	0.751	0.997
10			0.003	0.022	0.067	0.142	0.248	0.384	0.548	0.737
11			0.001	0.012	0.040	0.094	0.172	0.276	0.403	0.554
12			0.001	0.006	0.024	0.062	0.120	0.200	0.301	0.421
13				0.003	0.015	0.040	0.084	0.145	0.225	0.321
14				0.002	0.009	0.026	0.058	0.106	0.169	0.248
15				0.001	0.005	0.017	0.040	0.077	0.127	0.191
16					0.003	0.011	0.028	0.056	0.096	0.148
17					0.002	0.007	0.019	0.040	0.072	0.114
18					0.001	0.004	0.013	0.029	0.053	0.088
19						0.003	0.009	0.021	0.040	0.068
20						0.002	0.006	0.015	0.030	0.052
21						0.001	0.004	0.010	0.022	0.040
22						0.001	0.003	0.007	0.016	0.031
23							0.002	0.005	0.012	0.024
24							0.001	0.004	0.009	0.018
25							0.001	0.002	0.006	0.014
26								0.002	0.005	0.010
27								0.001	0.003	0.008
28								0.001	0.002	0.006
29								0.001	0.002	0.004
30									0.001	0.003
31									0.001	0.002
32									0.001	0.002
33										0.001
34										0.001
35										0.001

7. A PROOF OF MAXWELL'S LAW, THE PRINCIPAL PROPOSITION IN THE KINETIC THEORY OF GASES

First published in "Fysisk Tidsskrift" Vol. 23, 1925, p. 40.

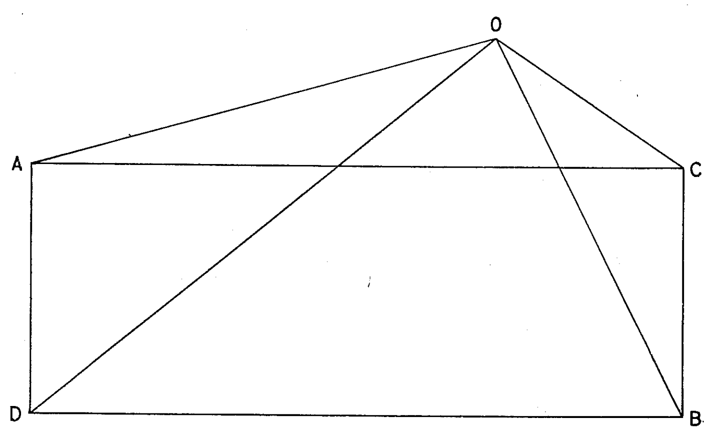
1. In the sixties of the nineteenth century, the old inconsistent and vague notions of the molecules of the gases were superseded by a positive theory *(Maxwell, Boltzmann)*, and the head stone of this theory is the Maxwellian Law of Distribution. This came to be the initiation of an entirely new view of, and working method in Physics, *viz.* the view and method based upon the Theory of Probabilities generally known, of late years, as the "Statistical Method". For many years it had to face the misapprehension, indifference, and antagonism of a mighty school oỵ natural scientists; but then one must admit that this aversion has been done away with in the present century, and the new methods are now, undoubtedly, recognized as the only serviceable with respect to a steadily growing number of phenomena, including the electrical. It is now rather obvious that the old manner of dealing with problems such as the principles of the Theory of Heat is not satisfactory, and that paths leading to better comprehension can be found. A concurrent of the change that has occurred was, of course, the experimental investigations that gradually banished any doubt as to the actual existence of the molecules as individuals, and their possession — to some extent, at least — of the properties upon which the theories were based (*Christiansen, Knudsen, Perrin*, and others). It is a matter of course that this development is not only physically significant; it must interest, also, those who work with the Theory of Probabilities and its applications on the whole, as indicated by the literature of recent years.

It admits of no doubt whatever that many people must have wanted to partake of the understanding thus attained. One might, perhaps, expect to find in the more recent manuals on physics, if not a thorough discussion of problems belonging here (such as diffusion, currents in gases, thermal conduction, the two specific heats) — which questions, by the way, have as yet hardly been treated in a quite unchallengeable manner, on the given presuppositions —, then at least a simple and clear proof of the Maxwellian law of distribution. Any such proof is seldom or never to be found, however; even special works on the Kinetic Theory are of no avail in this respect. I have therefore put down in writing the following proof which is based upon the very simplest and most well-known mathematical and mechanical theorems (I have not even used, *e. g.*, the Principle of Energy); consequentially, I believe, the characteristics of the reasoning stand a better chance of being recognized.

2. *Presuppositions about the Nature of the Molecules and the Effects of the Collisions.*

The molecules are all alike; they are hard, smooth, elastic spheres; and they are homogeneous or, if you like, their distribution of mass is mechanically equivalent to that of a homogeneous body. They move in a large space, with constant velocity along straight lines so long as no collisions occur. As usual, the velocity may be represented by a vector going out from the centre of the sphere. We make no special presuppositions as to the size of the spheres in proportion to the distances. It may be advantageous to replace the three-dimensional domain (space) which we are dealing with, strictly speaking, with a two-dimensional one (a plane); this would make the drawing of figures and the formulating of the proof slightly easier, although everything essential will remain the same in both cases.

The laws governing collision of elastic spheres are well-known; already *Huygens* set forth the main principles, though only for the uni-dimensional problem of central impact. Let us assume that we know the two velocities OA and OB before the impact; at this occasion it will be convenient to plot them from the point of contact O of the spheres. Further, we know the direction of thrust: the spheres being smooth, this is identical with



the radii to the point of contact at the moment of the impact. The impact will change OA to OC, and OB to OD. For the determination of C and D we have: AC is equal to, and parallel with, BD (but of opposite direction), and both are parallel with the direction of thrust; further, the quadrangle $ACBD$ is (not only a parallelogram, but also) a rectangle. That this is the case can be proved as follows: — During the first period of the elastic impact, *i. e.* while the compression is increasing, the end of the one vector will move from A to M, M being the middle point of AC; likewise, the end of the other vector will move from B to N, N being the middle point of BD. The relative velocity before the impact was AB; now it is MN. But this first period will cease when the relative velocity is at right angles to the direction of thrust; then the second period begins during which the spheres move away from each other, and it lasts until the state of contact is discontinued.

It should be remembered that no rotation is involved, according to the presuppositions; either the spheres do not rotate at all, or else the rotations will remain unchanged even during the collision.

3. *Concerning the Initial State.*

We will suppose that the state of the molecules (*i. e.* their position and velocity) at a certain time is as follows. The molecules are assumed to be distributed quite accidentally in the plane (or, strictly speaking: space) concerned, independent of each other, with the one exception that no two molecules can occupy the same place partly or wholly. As mentioned before, the velocity of an arbitrary molecule shall be represented by a vector going out from the molecule. We can consider an infinitely small area (strictly speaking, a volume) around a point situated at a distance r from the molecule in some direction or other. We let the probability of a velocity, the vector of which has its terminal point in this small area, be proportional to e^{-kr^2} and to the magnitude of the area but independent of the direction. Another, more exact way of expressing the same is to say that the probability density in this portion of the plane and e^{-kr^2} are in proportion. The probability density (also sometimes called the point probability) is defined as the limit value of the ratio between the probability corresponding to a certain area and the magnitude of the said area as both converge towards zero. The precise "physical" significance of the quantity k need not be considered here. — The state thus described may be called the normal or Maxwellian state. Now the problem is to prove that this state is in "statistical equilibrium", *i. e.* that conditions as a whole will not cease to be as described above, in spite of the changes in the separate molecules caused partly by the rectilinear motion, partly by the collisions. For, having proved this, we shall also have proved Maxwell's law.

4. The Proof.

To begin with, it is obvious that the usual rectilinear motion will not discontinue the Maxwellian state. Let us now consider a collision which is to take place in the point O. The initial velocities are $OA = a$ and $OB = b$, and the direction of thrust is given.

Now, we want to find the probability (more exactly, the probability density) that a collision of this kind will occur in the near future; later, we shall have to let both A and B traverse the entire plane. Pursuant to the preceding, and in accordance with the theorem about multiplication of probabilities, the expression for the sought-after probability must contain, as factors, firstly the expression

$$e^{-ka^2} \cdot e^{-kb^2} = e^{-k\,(a^2 + b^2)};$$

secondly, the relative or reciprocal velocity $AB = p$; and finally, cosine of the angle BAC between the said velocity and the direction of thrust; for, when the spheres are rather near to each other — which is the only case we need consider — and moving towards each other, $p \cdot \cos \angle BAC$ will be the velocity with which the spheres are approaching each other. Consequently, this quantity will express the permissible distance between the spheres if we make it a condition that the collision take place within a certain, narrow time limit.

We will now pass on to a consideration of the reversed collision, *i. e.* the direction of thrust is the same as before but the initial velocities are OC and OD, and — as is at once obvious — the final velocities are OA and OB; we can here use the same drawing as before. In order to prove that the two probabilities are identical, we use the three equations:—

$$a^2 + b^2 = c^2 + d^2 \qquad (1)$$

$$AB = CD \qquad (2)$$

$$\angle BAC = \angle CDB \qquad (3)$$

(2) and (3) are at once obvious. The equation (1) is obtained by a simple elementary-geometrical demonstration (an auxiliary line from O at right angles to the two sides of the rectangle; the Pythagorean proposition); it might also be obtained from the principle of energy, but that would evidently be making a detour.

We will now divide the whole plane into small squares (or, if you like, rectangles), the sides of which are partly parallel to, and partly at right angles to the direction of thrust. If A is situated inside such a square and B inside another, then C and D will be situated inside a similar pair of squares. We have now proved the existence of statistical equilibrium although only as far as the impacts hereto corresponding are concerned. But, as we can now proceed to choose a new pair of squares for A and B, and so on, and so forth, until there are no more possibilities, we have herewith proved Maxwell's theorem.

8. HOW TO REDUCE TO A MINIMUM THE MEAN ERROR OF TABLES

First published in "The Napier Tercentenary Memorial Volume", 1915, p. 345.

In the years which have passed since the memorable invention of logarithms, different systems have been devised for the arrangement and calculation of tables of logarithms, chiefly for the purpose of combining rapidity and convenience in use with a considerable degree of accuracy. Some of these systems will be discussed in the following remarks.

1. Firstly, we will consider the simple type of tables, which are intended for ordinary linear interpolation. Sometimes the first-differences are directly given, and, as a rule, tables of proportional parts are provided, while economy of space is effected by the double-entry arrangement (already used by *J. Newton* in the seventeenth century). For the sake of brevity this type of tables will in the following be referred to as "Type A". They are, even nowadays, preferred by many calculators, although other types (see below) undoubtedly surpass them as to convenience and rapidity.

Now let us consider accuracy. We may here suppose that the higher differences are unimportant, or the function nearly linear. It is here essential to give, not the maximal error, nor the probability of some special value of the error, but the mean-square of errors (which is the square of the mean-error, according to the general significance of this expression in the theory of probabilities). As unit of the error we will use the last decimal unit of the table-values. If we consider only the values directly printed, the mean-square of errors is known to be 1/12. If we suppose the interpolations distributed evenly along the whole table-interval, we find the mean-square of errors to be

$$1/18 + 1/12 = 5/36, \text{ or } 0{\cdot}1388 \ldots.$$

If, however, only the points dividing the interval into 10 equal parts are considered, we find a slightly lower value,

$$\frac{403}{3000}, \text{ or } 0{\cdot}13433 \ldots\ldots$$

It is not necessary to give here the exact law for the distribution of the errors, which is, furthermore, not very different from the "normal" law of errors; the reader interested in this question should consult *De accuratione qua possit quantitas per tabulas determinari*, by *Carolus Æmilius Mundt*, Havniæ, 1842.

2. We will now consider another well-known type, which will hereafter be referred to as "Type B". The ordinary double-entry table is here accompanied by a special auxiliary table, the separate horizontal lines of which correspond to the horizontal lines of the main table. By means of the values of the auxiliary table, which are simply 1, 2, 3 ... 9-tenths of the mean-difference of the opposite part of the main-table, all the interpolations necessary are reduced to simple additions. This convenient form of table seems to have been used not earlier than in the nineteenth century, and the oldest tables, as far as I know, are the following two: *Fünfstellige Logarithmen*, by *A. M. Nell* (1866), and *Logarithms and Antilogarithms*, issued by the Institute of Actuaries in 1877 (4 figures). The arrangement used in some parts of *Tables trigonométriques décimales*, by *Borda* and *Delambre* (1801) is, however, essentially the same. Obviously some modifications of this type of table are possible; but the arrangement described here is the most natural, because the number 10 is the basis of the system of numeration.

Considering accuracy, we find the mean-square of the error to be

$$1/6 \text{ or } 0.1666 \ldots\ldots$$

3. An increase of accuracy, without loss of convenience, has been accomplished by the appearance of *Fircifret Logaritmetabel*, by *N. E. Lomholt*, 1897 (the first edition). This author's object has been to do away with the great errors (exceeding 1·05 units), and furthermore to diminish the number of errors exceeding 0·5, as well as to diminish the "average error". He has not, as sometimes stated (for example, in the *Mathematical Encyclopædia*, both editions), reduced the average error to a minimum, and he has not thought it necessary to use a definite method, excluding entirely the personal element. Although, in my opinion at least, this is a drawback, I willingly acknowledge not only the real progress made, but also the general idea of improving the tables of Type B by means of new values both in the main table and in the auxiliary table.

4. In a Danish paper, published in *Nyt Tidsskrift for Matematik*, 1910, the present writer proposed to choose the 10 + 9 or 19 values of each horizontal line in such a manner that the sum of the square of the 100 resulting errors will be reduced to a minimum. It is hardly necessary to

say anything here in defence of this proposal[1]); but I will shortly set forth the method by which the 19 values may be found, using as an example the calculation of a single line (61) of the 4-figure table of logarithms.

Let us, for the present at least, accept the values of a 7-figure table as exact. As a starting-point we shall use a preliminary set of 19 values taken from a table of Type B. From these we derive 100 preliminary values of logarithms, and we write them down in form of a square. The 100 values are subtracted from the 100 corresponding "exact" values, and the 100 differences (or "errors") are written down in a similar manner. As some of the differences are negative, it is convenient to use either the number 9 or, still better (in accordance with the late Professor *T. N. Thiele*'s suggestion), the letter ν as representing a negative unit prefixed to a number, the following decimal parts being positive. By adding the vertical and the horizontal arrays we can obtain the sums V and H. These sums we arrange according to magnitude in two parallel columns, the former to the left, with the 10 terms decreasing from the top downwards, the latter to the right, with the 10 terms increasing from the top downwards. (Practically we always find that the difference between two arbitrary terms of each column will be less than 10, numerically; if not, we use another preliminary set of values satisfying this condition). We further extend the two columns downward by the repetition of some of the terms at the top, having previously subtracted 10 from each of the values V and added 10 to each of the values H.

If now we wish to alter the preliminary table in order to obtain the final table, we may represent the resulting changes in the 100 values of logarithms by following a certain number (m) of the vertical arrays and a certain number (n) of the horizontal arrays, adding 1 to the values of

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[1]) In: *Nyt Tidsskrift for Matematik B*, vol. 22, 1911, p. 10, Erlang states the following reasons why the method of least squares should be used: "In practice, the errors contained in any particular table do not manifest themselves immediately or one by one. The numbers corresponding to the logarithms to be looked up in the table — usually the results of measurements — contain errors already; several logarithms must be added to, or subtracted from, other logarithms in order to find the logarithm of the number that is to be looked up in the table of antilogarithms. The detrimental effect of the resultant error is some function of the magnitude of the error. This function cannot, as a rule, be determined; but then, this is fortunately not necessary. The probability that the resultant error exceeds a certain quantity x depends only on the ratio of x to the resultant mean error and decreases as the latter decreases; but this resultant mean error can be expressed in a well-known and very simple manner by the mean errors corresponding to the different sources of errors. These mean errors should therefore be minimum. It is assumed in the above that the resultant errors satisfy the typical, or exponential, law of errors; that this is actually the case can be proved, however, even without knowledge of the particular laws of errors applying to the different error sources in question if only the number of these sources is fairly great."

Log 6100 ... 6199.

	(+)	(+)									
	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
(—) 1	4	1	9	6	3	0	7	4	1	8	
1	4	1	9	6	3	0	7	4	1	8	
2	5	2	0	7	4	1	8	5	2	9	
(—) 3	6	3	1	8	5	2	9	6	3	0	
(—) 4	7	4	2	9	6	3	0	7	4	1	
4	7	4	2	9	6	3	0	7	4	1	
5	8	5	3	0	7	4	1	8	5	2	
(—) 6	9	6	4	1	8	5	2	9	6	3	
6	9	6	4	1	8	5	2	9	6	3	
	298	412	ν514	ν605	ν684	ν751	ν807	ν852	ν885	ν906	ν8714
	010	123	ν224	ν313	ν391	ν457	ν512	ν555	ν587	ν608	ν5780 (+)
	722	833	ν933	021	098	163	217	259	290	309	2845
	434	544	ν643	ν730	ν805	ν869	ν922	ν963	ν992	011	ν9913
	145	254	ν352	ν438	ν512	ν575	ν626	ν666	ν695	ν712	ν6975 (+)
	ν857	ν965	ν061	ν146	ν219	ν281	ν331	ν370	ν397	ν413	ν4040 (+)
	568	675	ν770	ν854	ν926	ν986	035	073	099	114	1100
	279	385	ν479	ν561	ν632	ν692	ν739	ν776	ν801	ν815	ν8159
	ν990	095	ν188	ν269	ν339	ν397	ν444	ν479	ν503	ν516	ν5220 (+)
	701	805	ν896	ν976	045	102	148	182	205	216	2276
	3004	4091	ν5060	ν5913	ν6651	ν7273	ν7781	ν8175	ν8454	ν8620	
	(—)	(—)									

7·095
ν82·015

25·080 — 22 = 3·080

2·344
ν78·801

23·543 — 25 = ν8·543

97·398
ν79·901

17.497 — 19 = ν8.497

p	V	H	*q*
ν5			5
	4091	ν4040	
ν6			4
	3004	ν5220	
ν7			3
	ν8620	ν5780	
ν8			2
	ν8454	ν6975	
ν9			1
	ν8175	ν8159	
0			0
	ν7781	ν8714	
1			ν9
	ν7273	ν9913	
2			ν8
	ν6651	1100	
3			ν7
	ν5913	2276	
4			ν6
	ν5060	2845	
5			ν5
	ν4091	4040	
6			ν4
	ν3004	5220	
			ν3
		5780	
			ν2
		6975	

the former and subtracting 1 from the values of the latter. In the corresponding arrays of the set of 100 errors corresponding changes will take place, subtractions in the vertical and additions in the horizontal arrays. The resulting improvement (the diminution of the sum of the squares of errors) we can easily find by the formula:

$$I = 2\,[V] - 2\,[H] - 10\,m - 10\,n + 2\,mn,$$

[V] being the sum of the m quantities V, and [H] the sum of the n quantities H. It is now at once obvious that in order to find these quantities in the columns, we must proceed from the top of each column downwards to a certain point, without skipping over any terms. Thus the only problem left is to find the points where we are to stop, or, in other words, the numbers m and n. But if we are to stop between V_m and V_{m+1}, and between H_n and H_{n+1}, the following conditions —

$$V_m > 5 - n > V_{m+1}$$

$$H_n < m - 5 < H_{n+1}$$

— must be satisfied, as, otherwise, it would be better to take one step more or less, down the right or the left column. If we use the numbers

$$p = m - 5$$

$$q = 5 - n$$

instead of m and n for the numeration of the successive intervals of the columns H and V (see example), the two conditions and also the expression of I take the simple forms:

$$V_m > q > V_{m+1}$$

$$H_n < p < H_{n+1}$$

$$I/2 = [V] - [H] - (pq + 25).$$

It is now easy to find out the only pairs of values of p and q compatible with the two conditions (giving what may be called the relative maxima of I). The result may be marked by limiting lines, as shown in the example. For the final choice we must, by means of the formula, calculate the corresponding values of I, ordinarily very few in number, taking the values $pq + 25$ from a small table with two entries. The greatest value of I/2 is chosen. The signs (+) and (—) indicate the resulting deviations from the preliminary table.

It happens in a few cases that the number of decimal places (here 7) used in the calculation turns out to be insufficient, but it is easy to give a rule covering these cases.

If this sort of calculation is to be undertaken on a large scale it is best to try to get rid of part of the work, for example, by finding the sum of 10 function-values with equidistant arguments, without actually undertaking the addition. For special functions, such as antilogarithms, the way to proceed is obvious. Furthermore, the function considered in most cases is so nearly linear that we can find the mean of the 10 values from the value corresponding to the mean-argument, applying, if necessary, a small and easily determinable correction.

A set of 4-figure tables[1]) of the type described, has been calculated by *H. C. Nybølle*, Mathematical Assistant at the Danish Statistical Department, and myself. Another collection also containing 5-figure tables is at present being elaborated.

I would only like to mention that similar principles might possibly find application for the construction of tables of some simple and practically important functions of complex variables.

5. Concerning the mean-square of errors in tables of Type B_1, I have tried unsuccessfully to find the exact value as in the cases A and B (see above), especially for the purpose of finding the difference in this respect between B and B_1. The solution of this problem is theoretically possible, under the same supposition as to the nature of the function as above, and the integrations necessary are very simple; but the number of cases to be considered is very great. Some indications may, however, be had from the experiences available; thus we find, that the improvement I (for a set of 100 values) is, on the average, about 2 or 3 units; sometimes, although seldom, it will be as great as 50 (about), sometimes 0. We might also consider the case in which the second differences of the function are considerable (although one can, of course, get rid of this case by altering the interval). In this case the mean-square of errors produced by the aforesaid cause will obviously be about four times greater for Type B than for Type B_1.

It seems probable that Types B or B_1 will be much used in the future for the construction of tables of different functions, and if stress is laid on the greatest accuracy compatible with the arrangement, space, and number of figures chosen, Type B_1 should be preferred.

[1]) *A. K. Erlang, Fircifrede Logaritmetavler,* G. E. C. Gad, København. (1910-11), three editions, A, B, C, the last being the most complete.

9. AN ELEMENTARY TREATISE ON THE MAIN POINTS OF THE THEORY OF TELEPHONE CABLES

First published in "Elektroteknikeren", Vol. 7, 1911, p. 139.

It is undoubtedly a matter of common knowledge that the progress experienced of late years with regard to long distance telephony is chiefly attributable to the introduction of new types of cable with artificially increased self-inductance, and that the difficulties deriving from the considerable capacitance between the conductors of paper core or gutta-percha cables have thereby been overcome, practically speaking. First and foremost, this important development is based on numerous theoretical investigations conducted, for instance, by *Heaviside* and Lord *Kelvin* in England, *Pupin* and later *Kenelly* in America, *Poincaré* in France, and *Breisig* in Germany. In Denmark, Prof. *Absalon Larsen* has — among others — taken part in this work. Among those who took the lead with respect to the construction of the new cables should be mentioned *Pupin* and, of the Danes, especially *J. L. W. V. Jensen*, Chief Engineer in the Copenhagen Telephone Company, and the late *C. E. Krarup*, department manager in the Danish State Telegraph Service. While the foreign literature of this subject is already very comprehensive, the Danish public has, presumably, so far only had opportunity to read a few articles which appeared in periodicals such as "Ingeniøren" (1903, 1911), "Fysisk Tidsskrift" (1902, 1904), and "Elektroteknikeren" (1909, 1911); the authors are *Krarup*, and the engineers *Walsøe* and *V. Clausen*. The articles contain, among other things, descriptions of constructions, results of speech tests, and calculations for the different types of cables; but they do not afford much guidance for anybody desirous of acquainting himself with the underlying theory, and only a limited number of people have, so far, heard *P. O. Pedersen*'s exposition of the subject in his lectures, of course. Under the circumstances the following dissertation, which originally was prepared as a part of the investigations of cable problems undertaken by the Telephone Company of Copenhagen, may perhaps be of some use as a basis for further study; an expert will hardly find much in it, as far as new results are concerned, but rather something in the line of simplified proofs and calculations. Many of the problems discussed are of consequence, as it happens, also for other electrotechnical domains than just telephony, for instance the conveyance of a. c. power electricity over long distances.

2. *Mathematical Basis* (*Vectors, Complex Numbers*).

In the following we shall be dealing exclusively with sinusoidal alternating currents and voltages, *i. e.* such alternating quantities, the variations of which can be represented by the projection of a straight line segment corresponding in length to the maximum value of the current (or voltage), and rotating with even velocity about one end-point O, on a fixed, straight line, the x-axis, going through the point O. The direction of rotation is supposed to be anti-clockwise. The frequency, or number of cycles per second, multiplied by 2π is called ω. If we are dealing with several simultaneous currents, or voltages respectively, all of the same frequency, we get — for each of these — a line segment issuing from O; we draw these lines in positions corresponding to definite, but otherwise arbitrarily chosen, points of time. The line segments are called vectors. It may be convenient, in certain cases, to let a vector issue from some other point than O, without altering its direction and length. Any vector can be represented, for algebraical manipulation, by a complex number or pair of numbers, in the form $a + ib$ where a denotes the projection on the x-axis, and b the projection on an y-axis normal to the x-axis. Now, any two vectors, or the corresponding complex numbers, can be added or multiplied, *i. e.* a new vector is produced by geometrical construction, or a new complex number is calculated, respectively, according to the following rules (*Caspar Wessel*, 1797): —

I. Addition. — Rule of construction: Let the two given vectors be represented by two adjacent sides of a parallelogram; the diagonal will then represent the sought vector. — From this is easily derived the rule of calculation:

$$(a + ib) + (c + id) = (a + c) + i\,(b + d),$$

where addends and augends are obviously interchangeable.

II. Multiplication. — Rule of construction: The angle between the x-axis and the sought vector must equal the sum of the angles between the x-axis and the two given vectors; the length of the sought vector, as measured in terms of the unit employed, must equal the product of the lengths of the two given vectors. — Hence it follows that the factors are interchangeable, and that the theorem

$$(x + y)\,z = xz + yz$$

holds good also for complex numbers, as is easily proved by means of two similar triangles. — Now the rule of calculation can be derived:

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

It should further be noticed that the rule of multiplication especially implies that

$$(a + ib)^2 = a^2 - b^2 + i \cdot 2ab;$$

we shall also need the following formula for extraction of square roots:

$$\sqrt{g + ih} = \sqrt{\frac{k + g}{2}} + i\sqrt{\frac{k - g}{2}},$$

where the terms on the right side are to be taken with like signs for h positive, but with unlike signs for h negative, and where

$$k = +\sqrt{g^2 + h^2};$$

the formula is easily proved by raising the right side to the second power.

The accuracy of the slide rule will not, as a rule, be sufficiently great for practical calculations with complex numbers; logarithms should therefore be used, and preferably also a table of squares. (A few remarks of mathematical nature will be postponed until they can be connected).

3. *Physical Basis (Ohm's Law for Alternating Currents).*

It has been proved by experience that the different types of conductors employed in practice for the purpose of carrying alternating current, all have certain simple properties: when the voltage is sinusoidal and has a certain frequency, the current will also be sinusoidal and have the same frequency, and vice versa; this is true, however, only when conditions have become stationary. We shall therefore concern ourselves with what happens under stationary conditions only. Ohm's law, according to which the voltage is equal to the product of current and resistance, applies here, too, the resistance being a complex quantity just as voltage and current are complex quantities; the multiplication is carried out in accordance with the rule stated above. In other words, when different a. c. voltages are applied to one and the same conductor, the ratio of the maximum values of voltage and current and the phase difference will always be the same. It is often preferred to manipulate with conductance instead of resistance; conductance being the reciprocal of resistance, it is necessary to adapt Ohm's law accordingly. Starting from the usual definitions of self-induction and capacitance, it is easy to show that an inductance of L henry represents a resistance

$$i\omega L,$$

and that a capacitor with a capacitance of C farad has a conductance

$$i\omega C.$$

For the sake of brevity, only the first of the two propositions shall be proved here; the second can be proved in a similar manner.

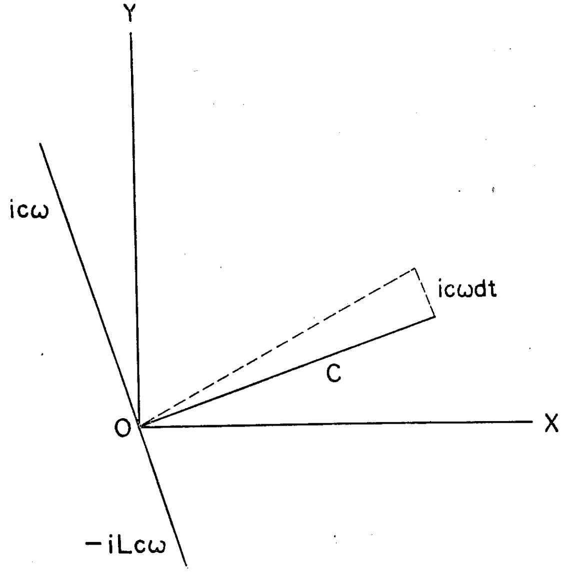

Fig. 1.

In fig. 1, the vector c represents the current; in an infinitely small space of time dt during the rotation of the vector, its end-point will cover a distance the actual length of which is $c\omega dt$, but nevertheless must be written

$$ic\omega dt,$$

because the route is normal to c. As it will appear by projection on the x-axis, the vector $ic\omega dt$ will always during the rotation represent the increment of the current for the time dt; the vector $ic\omega$ will, accordingly, represent the velocity of this increment. The self-induced e. m. f. is represented by the vector $-iLc\omega$, that is the inductance acts as a resistance $iL\omega$.

The reader can now, by means of what was said above respecting calculations with complex numbers, ascertain that problems concerning conductors connected in parallel or in series can be treated as d. c. problems.

When, in the following, the words resistance or conductance are used, it must be remembered that they always refer to an alternating current of this or that frequency; similarly, when such quantities are designated by ordinary small letters, the latter nevertheless stand for complex numbers.

4. *The Infinitely Long Homogeneous Cable* (*Near-End Resistance*).

We shall now proceed to investigate an infinitely long cable, partly because the question of long distance telephony is particularly important, and partly because we shall obtain certain results which will be useful later.

First, we will consider a circuit (see fig. 2) consisting of two conductors, with the resistance r_1 and a leakance s_1 repeated alternately to infinity.

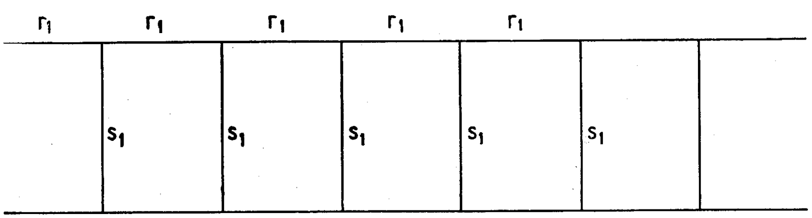

Fig. 2.

It is unimportant in what proportion the resistances r_1 are actually divided between the two wires; it is therefore unnecessary to consider the case of a single wire circuit separately. We will find the total or loop resistance z_1, measured at the near-end of the two-wire circuit. We get:

$$r_1 + \cfrac{1}{s_1 + \cfrac{1}{z_1}} = z_1, \tag{1}$$

or,

$$\frac{z_1 (r_1 s_1 + 1) + r_1}{z_1 s_1 + 1} = z_1, \tag{2}$$

from which z_1 can be found easily. The special case of r_1 and s_1 being infinitely small permits omission of $r_1 s_1$, which gives

$$z_1 = \sqrt{\frac{r_1}{s_1}} \tag{3}$$

Now this can be applied to an infinitely long homogeneous cable having the resistance r, and the leakance s per unit length. Then, imagining that the unit length has been divided into m portions, we put $\frac{1}{m} \cdot r$ instead of r_1, and $\frac{1}{m} \cdot s$ instead of s_1. Taking $m = \infty$ we find

$$z = \sqrt{\frac{r}{s}} \tag{4}$$

5. The Infinitely Long Homogeneous Cable (Transmission).

We have a circuit consisting of a resistance r_1 and a leakance s_1; at the receiving end is added a resistance z_1 (fig. 3)

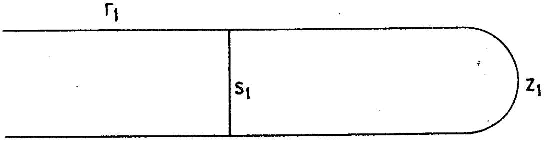

Fig. 3.

Both current and voltage will be attenuated as a result of being conveyed by this circuit, the former by passing s_1 and the latter by passing r_1. The ratio of the current values on the right and left side of s_1 is

$$\frac{1}{z_1} : \left(s_1 + \frac{1}{z_1}\right)$$

The ratio of the voltage values on the right and left side of r_1 is

$$\frac{1}{s_1 + \frac{1}{z_1}} : \left(r_1 + \frac{1}{s_1 + \frac{1}{z_1}}\right).$$

The two ratios are equal, since the last parenthesis is equal to z_1, according to (1). This was to be expected as the ratio of voltage to current must be the same on the left and right side of r_1 and s_1.

In the special case of r_1 and s_1 being infinitely small, the found expressions will read

$$\frac{1}{1 + \sqrt{r_1 s_1}}.$$

We will now consider a unit length of the previously mentioned, infinitely long homogeneous cable, and we divide this unit length into m portions. In order to find the ratio of the currents transmitted and received by each unit length, we must seek the limit, as m tends to infinity, of

$$\left(\frac{1}{1+\frac{1}{m}\sqrt{rs}}\right)^m.$$

Transforming $\sqrt{rs}$ into $\beta + i\alpha$ where β and α are real numbers, we have

$$\left(\frac{1}{1+\frac{1}{m}\sqrt{rs}}\right)^m = \left(\frac{1}{1+\frac{1}{m}\beta}\right)^m \cdot \left(\frac{1}{1+\frac{1}{m}i\alpha}\right)^m.$$

We find that the limit of the first factor is evidently

$$e^{-\beta},$$

e being the limiting value of $\left(1+\frac{1}{x}\right)^x$ as x, which is real, approaches infinity; e has the value 2·718 to three places of decimals. — As to the second factor can be mentioned that the infinitely small vector $\frac{1}{m}\,i\alpha$ is normal to the vector 1 which, therefore, neither increases nor diminishes by the addition; it only turns through a certain angle which, measured by the length of the arc it subtends in a circle with radius 1, is $\frac{1}{m}\,\alpha$. The sought limit, then, is a vector of the length 1 turned at an angle $-\alpha$ to the x-axis. (The proof outlined here can easily be given an exact form). Expressed in degrees, the angle is $\frac{1}{m}\alpha\cdot\frac{180}{\pi}$. We have now determined the attenuation and phase displacement of the current (and voltage) for the unit length of cable. It is often preferred to let the factor $e^{-\gamma}$, where

$$\gamma = \beta + i\alpha = \sqrt{rs}, \tag{5}$$

denote both these changes together; the expression $e^{\beta+i\alpha}$ designates a vector of the length e^{β} forming an angle α with the x-axis, and it can be shown that the usual rules for calculations with numbers raised to real powers are valid also in the case of complex powers.

It should be noticed that the ratio of attenuation of energy must be $e^{-2\beta}$, since current and voltage both have the attenuation ratio $e^{-\beta}$.

It we consider a cable of the length l instead of 1, we get the values βl, al, γl instead of β, a, γ; γl is called the transmission coefficient.

The numerical calculation of β and a is not particularly difficult; as

$$r = R + i\omega L, \qquad (6)$$

$$s = S + i\omega C, \qquad (7)$$

where R, L, C, S are real numbers, we obtain:

$$2\,\beta^2 = \sqrt{(\omega^2 L^2 + R^2)\ (\omega^2 C^2 + S^2)} - (\omega^2 CL - SR) \qquad (8)$$

$$2\,a^2 = \sqrt{(\omega^2 L^2 + R^2)\ (\omega^2 C^2 + S^2)} + (\omega^2 CL - SR) \qquad (9)$$

It is generally assumed that L, R, C, S retain the same values for all frequencies, but that is not strictly correct.

6. Continuously-Loaded Cables.

The method of spinning iron wire over the conductors in a cable was first put to practical use by *Krarup* (1902), since when it has found rather extensive employment. A few years before, *J. L. W. V. Jensen* had made a suggestion of the same nature and experimented with a very similar type of cable. The idea of improving the cables by increasing their

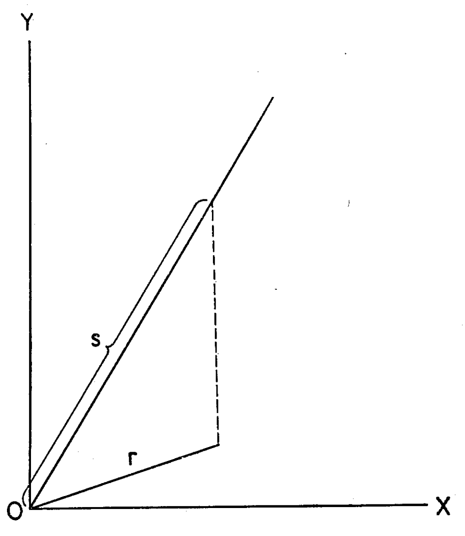

Fig. 4.

self-induction can be traced, in a more vague form, to a still earlier date. The effect of the iron wire upon the conductor is chiefly that of increasing the inductance; we shall here assume that this does not involve a simultaneous change of the values of the other constants. We want to find the optimum value of L as specified by the condition that an infinitely small variation of L, *i. e.* an infinitely small variation dr of r in the direction of the y-axis, must cause an infinitely small variation $d\gamma$ of γ, also in the direction of the y-axis. We have:

$$\gamma^2 = r \cdot s \tag{10}$$

$$2\,\gamma d\gamma = s \cdot dr; \tag{11}$$

accordingly, γ and s must have the same direction, and again, r and s must have the same direction. The problem is solved. For various reasons, however, it is difficult to get so much inductance in practice.

7. *Kennelly's Two Diagrams. — The Principal Constants.*

Fig. 5 shows the first Kennellian diagram, the T-network, where a is

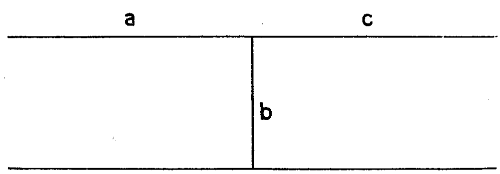

Fig. 5.

a resistance, b a leakance, and c a resistance. It has the following 3 principal properties when an additional arbitrary resistance x is connected in series with c:

The ratio of transmitted to received current is

$$\frac{\frac{1}{b}}{\frac{1}{b} + c + x};$$

The ratio of transmitted to received voltage is

$$\frac{x}{x\,(ab + 1) + abc + a + c};$$

The total resistance is

$$\frac{x\,(ab + 1) + abc + a + c}{xb + bc + 1}.$$

These properties — of which any two lead to the third — and only these are significant for the performance of this connexion as a cable. We will now introduce 4 constants p, q, u, and v, being defined as follows:

$$\left.\begin{aligned} p &= ab + 1 \\ q &= abc + a + c \\ u &= b \\ v &= bc + 1 \end{aligned}\right\} \quad (12)$$

Of these, p and v are abstract numbers, q is a resistance value, and u is a conductance value. As it will appear from the following, it is justifiable to call them "the principal constants".

The 4 constants count, as a matter of fact, for 3 only, because we have

$$pv - uq = 1 \quad (13)$$

It is easy to see how a, b, and c can be expressed in terms of the principal constants; in practice, of course, the real parts of a, b, c must be positive; this is not necessary, on the other hand, when the above notations are used exclusively as a means of help in the calculations. We can now write new and simple expressions for the current ratio, the voltage ratio, and the total resistance, respectively, *viz.:*

$$\frac{1}{v + ux},$$

$$\frac{x}{px + q},$$

$$\frac{px + q}{ux + v}.$$

The relative positions of the two ends of the two-wire circuit can be reversed by interchanging p and v.

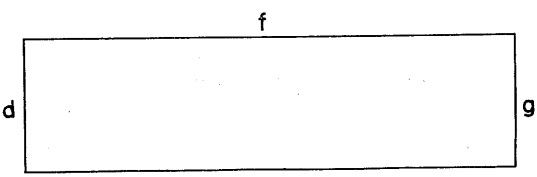

Fig. 6.

The second Kennellian diagram, the π-network, is shown in fig. 6 where d is a leakance, f is a resistance, and g is a leakance.

When a conductance y is placed parallel to g, the current ratio, voltage ratio, and resultant conductance are, respectively,

$$\frac{y}{y\,(df+1)+dfg+d+g},$$

$$\frac{\frac{1}{f}}{\frac{1}{f}+g+y},$$

$$\frac{y\,(df+1)+dfg+d+g}{yf+fg+1}.$$

By introducing 4 "principal constants" as defined by

$$\left.\begin{aligned} p &= fg+1 \\ q &= f \\ u &= dfg+d+g \\ v &= df+1 \end{aligned}\right\}, \tag{14}$$

these expressions can be reduced to

$$\frac{y}{yv+u},$$

$$\frac{1}{p+qy},$$

$$\frac{yv+u}{yq+p}.$$

In the case of $y = \frac{1}{x}$, however, we find by comparison that the first two expressions are identical with the corresponding expressions for the T-network (see above), whereas the third (the conductance) is the reciprocal of the resistance as expressed above, assuming all along that p, q, u, and v are the same in both cases. We have now shown that a π-network can always replace a T-network, and *vice versa*, if only the principal constants are the same.

We will now consider a more complicated diagram consisting of any number of resistances and leakances (fig. 7). It is obvious without any calculations that this new diagram can be converted, step by step, into a T-network or a π-network by substituting, successively, π-networks for such portions of it as can be regarded as T-networks, or *vice versa*. If the total number of resistances and leakances is 4 or 5,

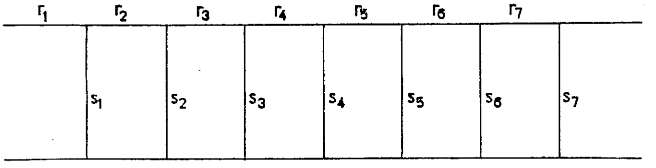

Fig. 7.

only 1 substitution is necessary; 2 substitutions are required if the total is 6 or 7, and so on. Thus, when resistance and leakance are distributed arbitrarily all along a length of cable, the cable can be described precisely by means of 4 principal constants.

8. *The Infinitely Long Periodic Cable.*

For an infinitely long cable consisting of an infinitely great number of cable segments each having the constants p, q, u, v, the total resistance x is given by the equation,

$$x = \frac{px + q}{ux + v}; \tag{15}$$

again, the current or voltage ratio for each segment is given by the equation,

$$e^{-\gamma l} = \frac{1}{v + ux}. \tag{16}$$

We obtain from these equations,

$$ux^2 + (v - p)\,x - q = 0 \tag{17}$$

and, because of (13),

$$e^{\gamma l} + e^{-\gamma l} = p + v. \tag{18}$$

In the special case of the cable segments being symmetrical, we have $p = v$; hence,

$$x^2 = \frac{q}{u}, \tag{19}$$

$$\left.\begin{aligned} e^{\gamma l} + e^{-\gamma l} &= 2\,p, \\ e^{\gamma l} - e^{-\gamma l} &= 2\sqrt{qu} \end{aligned}\right\} \tag{20}$$

9. The Finite Homogeneous Cable.

The last 2 results combined with the results from sections 4 and 5 enable us to determine the principal constants of the finite, homogeneous cable. We have,

$$\frac{q}{u} = \frac{r}{s} \tag{21}$$

$$e^{\sqrt{rs}\cdot l} + e^{-\sqrt{rs}\cdot l} = 2\,p \quad (\text{or, } e^{\sqrt{rs}\cdot l} - e^{-\sqrt{rs}\cdot l} = 2\sqrt{qu}); \tag{22}$$

accordingly,

$$p = v = \frac{1}{2}\left(e^{\sqrt{rs}\cdot l} + e^{-\sqrt{rs}\cdot l}\right) \tag{23}$$

$$q = \frac{1}{2}\sqrt{\frac{r}{s}}\left(e^{\sqrt{rs}\cdot l} - e^{-\sqrt{rs}\cdot l}\right) \tag{24}$$

$$u = \frac{1}{2}\sqrt{\frac{s}{r}}\left(e^{\sqrt{rs}\cdot l} - e^{-\sqrt{rs}\cdot l}\right) \tag{25}$$

10. The Connexion of Two (or more) Cables in Series.

The principal constants P, Q, U, V of the composite cable can be expressed, in terms of the principal constants p_1, q_1, u_1, v_1 and p_2, q_2, u_2, v_2 of the two parts, by the following simple equations:

$$\left.\begin{aligned} P &= p_1p_2 + q_1u_2 \\ Q &= p_1q_2 + q_1v_2 \\ U &= u_1p_2 + v_1u_2 \\ V &= u_1q_2 + v_1v_2 \end{aligned}\right\} \tag{26}$$

Taking the current ratio first, in order to prove (26), we find that we must have

$$\frac{1}{V + Ux_2} = \frac{1}{v_1 + u_1x_1}\cdot\frac{1}{v_2 + u_2x_2},$$

where x_2 is quite arbitrary, and

$$x_1 = \frac{p_2x_2 + q_2}{u_2x_2 + v_2}.$$

Hence we obtain the expressions for U and V. Taking the voltage ratio next, we get the expressions for P and Q.

From here it is easy to proceed to a consideration of a cable composed of more than two cable segments, which composite cable we may characterize as possessing properties corresponding to the "product" of its component segments, only the "factors" are not mutually interchangeable here; during the actual calculations, however, you are at liberty to form the product of nos. 1 and 2 first, or of nos. 2 and 3, and so on.

The applicability of the found results is rather widespread. They may be used in cases where telephone calls have to pass two or more channels of different nature connected together; they also apply to cases where capacitors, inductance coils, &c., are connected in series with or across the circuits, at the exchanges or at the subscribers. They apply, further, to the different types of transformers (toroidal transformers, induction coils, repeating coils, &c.); the ratios are here particularly simple, because the quantity u is purely imaginary. It should be noticed, however, that in the case of a transformer we are supposed to be dealing with a connexion of two circuits (primary and secondary) having lines of magnetic force in common, wholly or partly, but not otherwise influencing each other; and transformers of the types employed in practice will hardly fulfil such conditions completely.

11. Pupin Cables (Short Coil-Spacing).

The self-inductance of a cable can be increased by inserting identical coil inductances, or loading coils, in the cable at uniform intervals along the routes, as devised by *Pupin* in 1900.

We will assume for the present that the distance is so short and the amount of self-inductance so small that the inductance added can be regarded as evenly distributed. If we leave out of account the ohmic resistance added at the same time, we have the same conditions as in section 6. We will further assume, however, that the increment added to the original value of r consists of a real component and a purely imaginary component as well, the ratio of which is constant and known. We have,

$$\gamma^2 = r \cdot s,$$
$$2\,\gamma d\gamma = s \cdot dr;$$

accordingly, at the optimum point,

$$\gamma^2 \mathrel{+\!\!\!\!=} s^2 \cdot dr^2$$
$$rs \mathrel{+\!\!\!\!=} s^2 \cdot dr^2$$
$$r \mathrel{+\!\!\!\!=} s \cdot dr^2$$

where the symbol $\mathrel{+\!\!\!\!=}$ means: "has the same direction as".

The angle between s and r, then, must be twice the angle between the x-axis and dr. Hence follows a construction (a generalization of that in section 6) by means of which it is easy to decide exactly how much inductance should be added.

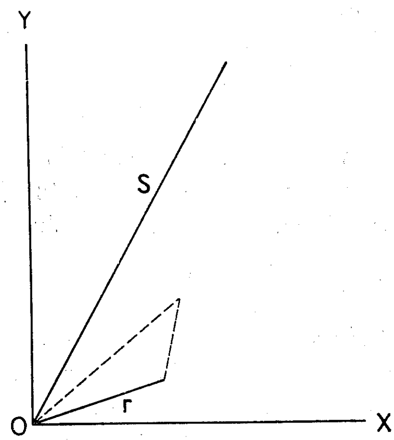

Fig. 8.

12. Pupin Cables In General.

We will now proceed to consider the case of an arbitrary, predetermined coil-spacing and investigate the effect of the Pupin coils inserted. Let us take a symmetric cable segment, having the constants

$$p_1, q_1, u_1, v_1,$$

together with one half of the preceding Pupin coil and one half of the succeeding Pupin coil, each half having the constants

$$1, q_2, 0, 1.$$

Now let the constants of the described circuit be called

$$p, q, u, v;$$

then, according to (26), we have

$$\left.\begin{aligned} p = v &= p_1 + u_1 q_2 \\ q &= q_1 + 2\, p_1 q_2 + u_1 q_2^2 \\ u &= u_1 \end{aligned}\right\} \qquad (27)$$

and, according to section 8,

$$e^{\phi} + e^{-\phi} = p + v \tag{28}$$

$$(e^{\phi} - e^{-\phi})^2 = 4\,qu, \tag{29}$$

where ϕ denotes the transmission coefficient.

From (27) and (28) follows that

$$\tfrac{1}{2}(e^{\phi} + e^{-\phi}) = p_1 + u_1 q_2 \tag{30}$$

This formula is of great importance because it serves to calculate the attenuation for any frequency when the constants of the cable and coils, and the coil-spacing are given. Tables of the values of

$$\tfrac{1}{2}(e^{x} + e^{-x}) \quad \text{and} \quad \tfrac{1}{2}(e^{x} - e^{-x})$$

for arbitrary values of x are required for the practical calculation of p_1, u_1, and ϕ; such tables as *C. Burrau:* Tafeln der Funktionen Cosinus und Sinus, or *G. F. Becker* and *C. E. van Orstrand:* Hyperbolic Functions, will be useful for this purpose, although they only give the values for real or purely imaginary values of x, but it is not difficult to form the values for arbitrary values of x by means of the tables. It is somewhat more difficult to find the quantity

$$\beta + i\alpha = \phi$$

from formula (30) which can be written in the form,

$$\tfrac{1}{2}(e^{\phi} + e^{-\phi}) = \xi + i\eta \tag{31}$$

where β, α, ξ, η denote real numbers. It is undoubtedly best to use the two formulae

$$\tfrac{1}{2}(e^{\beta} + e^{-\beta}) = \frac{r_2 + r_1}{2}, \text{ and} \tag{32}$$

$$\tfrac{1}{2}(e^{i\alpha} + e^{-i\alpha}) = \frac{r_2 - r_1}{2}, \tag{33}$$

where

$$r_2 = \sqrt{(\xi + 1)^2 + \eta^2} \tag{34}$$

$$r_1 = \sqrt{(\xi - 1)^2 + \eta^2} \tag{35}$$

the proof of which I shall omit for the sake of brevity.

13. The Determination of the Coefficient of Self-Induction for Pupin Coils.

The inductance added by inserting Pupin coils in a cable is never pure; the vector corresponding to the resistance q_2 forms a slightly acute angle with the x-axis, which angle in practice can be determined in advance. The value of q_2, on the other hand, is optional, and we will seek the optimum value, supposing the cable constants and coil-spacing to be given. When q_2 varies, ϕ will also vary, and we have according to (30) that

$$(e^{\phi} - e^{-\phi})\, d\phi = 2\, u dq_2, \tag{36}$$

and so, because of (29),

$$4\, qu \cdot (d\phi)^2 = 4\, u^2\, (dq_2)^2, \tag{37}$$

$$q\, (d\phi)^2 = u\, (dq_2)^2. \tag{38}$$

For the optimum point where $d\phi$ is parallel to the y-axis, we have

$$q \mp u\, (dq_2)^2.$$

By means of the formulae (27) this condition can be converted into

$$q_1 + 2\, p_1 q_2 + u_1 q_2{}^2 \mp u_1\, (dq_2)^2$$

We will presuppose that the direction of q_2 is given; accordingly,

$$dq_2 \mp q_2.$$

The condition is then

$$q_1 + 2\, p_1 q_2 \mp u_1\, (dq_2)^2 \mp u_1 q_2{}^2.$$

It is now easy to determine the sought value of q_2 by construction or calculation.

In the special case of q_2 being parallel to the y-axis, which simplifies our problem, we have for the optimum point

$$q \mp u.$$

The assumption that the cable segment is infinitely short will, of course, bring us back to the results from sections 6 and 11.

14. The Choice of Receiving Instruments.

When the type of cable to be used has been decided upon, the question of adapting the constants of the telephone instruments as well as possible to those of the cable suggests itself. Such instruments must be able to work both as transmitters and receivers at all times, which complicates the problem. We shall here restrict ourselves to the consideration of a transmitting instrument containing a microphone only, a receiving instrument containing a telephone receiver only, and a cable of such a length that it may be regarded as infinite. Since the space available for the coil windings of the telephone is limited, the resistance x of the receiver will be proportional to the second power of the number of turns, and the vector corresponding to x will have a given direction. In order to get a maximum number of ampere-turns, the numerical value of the product of the current and the square root of the resistance, *viz.*

$$\frac{1}{ux+v}\cdot\sqrt{x},$$

should be maximum; or the numerical value of

$$\sqrt{x}+\sqrt{\frac{r}{s}}\cdot\frac{1}{\sqrt{x}}$$

(see section 9) should be minimum. Putting μ instead of $\sqrt{x}$, we have for the optimum point, then,

$$\frac{1}{\mu}\left(\mu-\sqrt{\frac{r}{s}}\cdot\frac{1}{\mu}\right)d\mu \perp \mu+\sqrt{\frac{r}{s}}\cdot\frac{1}{\mu},$$

or, as $\mu \nparallel d\mu$,

$$\mu-\sqrt{\frac{r}{s}}\cdot\frac{1}{\mu} \perp \mu+\sqrt{\frac{r}{s}}\cdot\frac{1}{\mu},$$

$$x-\sqrt{\frac{r}{s}} \perp x+\sqrt{\frac{r}{s}}.$$

x is consequently determined by a circle where the end points of

$$\sqrt{\frac{r}{s}} \quad\text{and}\quad -\sqrt{\frac{r}{s}}$$

are diametrically opposite points; the resistance of the receiver must then be numerically equal to the resistance of the infinitely long cable.

In his lectures, *P. O. Pedersen* a few years ago stated the same result, only proved in a different manner. An investigation corresponding to the above could be carried out with similar means of help, even though the presuppositions are somewhat less simple than in the above.

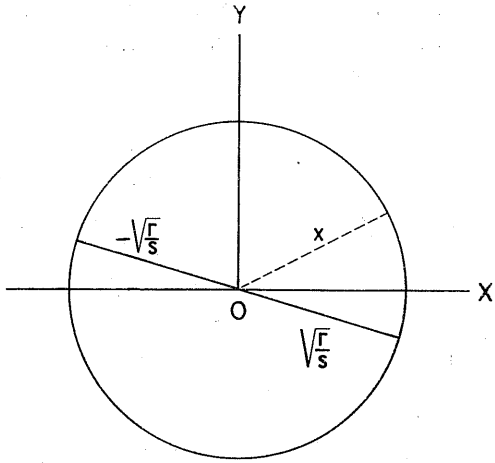

Fig. 9.

15. Methods of Measuring.

According to the foregoing, an investigation of a length of cable will rather be directed towards the determination of the 4 constants p, q, u, v; in consequence of the equation

$$pv - uq = 1$$

their number is actually reduced to 3, and in most cases it is even reduced to 2, on account of the symmetry. Apart from an alternating-current generator producing a current of the desired frequency, a suitable measuring instrument is required for the purpose mentioned, for instance a Wheatstone bridge with a telephone; it must comprise 3 known resistances, two of these being non-inductive and the third having a suitable, known inductance or capacitance. Balance can be obtained by adjusting the values of two of these, an unknown resistance being inserted as the fourth side of the quadrangle; the sought resistance is then easy to find. Thus it is possible to measure the resistance at either end of the cable, with the opposite end first short-circuited, and then open-circuited; in this manner the respective values of the ratios

$$\frac{p}{u}, \quad \frac{q}{v}, \quad \frac{v}{u}, \quad \frac{q}{p}$$

are found, which is more than sufficient for the determination of p, q, u, v. (An alternative method consists in connecting a resistance of known value across the opposite end of the cable during the measurements).

Professor *A. Larsen*'s "Complex Compensator" is just as convenient as the Wheatstone bridge, and its serviceability is more comprehensive. It was the subject of a lecture in the Danish Electrotechnical Association; the lecture has been printed in no. 25 of the journal of this association (1910), for which reason it is unnecessary to describe it here. Neither does the allotted space permit a description of another, somewhat simpler form of construction which it might otherwise, perhaps, be interesting to compare with Prof. Larsen's original construction. For the purpose of our present investigation, the compensator will accomplish the same as a Wheatstone bridge; p and v can furthermore conveniently be determined by measuring, partly, the potential difference at one end of the cable, and partly, the potential difference existing simultaneously at the other end. A pair of test wires may be used in the latter case, but their resistance need not be known.

Finally may be mentioned a recently suggested method for determining the transmission level and total resistance of the infinitely long cable by measuring an available cable segment. The method involves the use of a variable capacitor. A resistance, estimated to be equal to the resistance of the infinitely long cable, is added at one end. Control measurements at both ends of the cable will give identical results if the estimate is correct; if not, the resistance added must be altered in accordance with the results of the measurements, and so on. The final amount of resistance added is equal to the sought resistance. Next, the magnitude and direction of the quantity which in the above was given the symbol e^{ϕ}, can be found by measuring the potential difference at both ends of the cable segment. The compensator will undoubtedly be very suitable for this method.

16. Summary.

After a historical, mathematical and physical introduction (1, 2, 3) the principal formulae respecting the infinitely long homogeneous cable are deduced in a simple way (4, 5), the results being applied especially (6) to cables with artificial, evenly distributed self-inductance. It is shown (7) that any length of cable is characterized by 4, or actually 3, so-called principal constants; the connexion of these with the constants mentioned earlier is demonstrated (8, 9), and rules of calculation are given for the connexion of cables in series (10). Next are mentioned the coil-loaded Pupin cables (11, 12, 13) and the cooperation of the receiver with the circuit (14), and finally the application of the theory to the methods of measuring (15).

10. AN ELEMENTARY THEORETICAL STUDY OF THE INDUCTION COIL IN A SUBSCRIBER'S TELEPHONE APPARATUS

First published in "Elektroteknikeren", Vol. 10, 1914, p. 169.

1. Introduction.

An ordinary telephone instrument of the Local Battery type contains, in addition to a telephone receiver and a microphone with appurtenant battery, a transformer (toroidal induction coil, or plain induction coil) the primary winding of which is part of the local or microphone circuit, while the secondary is connected in series with the receiver and the external conductor. Even if these items are taken for granted, the description of the instrument would not be complete without mention of quite a lot of quantities of mechanical, electrical, or magnetical nature. In practice, a good many of these can be chosen at discretion, alone with a view to the greatest possible efficiency of the instrument. The most important problem in a theoretical investigation is therefore the determination of such values of the constants as will give the best possible result; it should be remembered, however, that in each particular case a certain latitude is permissible, having no perceptible effects in practice, not even by systematic speech tests.

2. Formulating the Problem; Denotations.

In order to find out which type of transformer is the best, we must know, firstly, the resistance r_1 of the microphone, and secondly, the characteristic impedance of the conductor, *i. e.* the apparent impedance of the infinitely long line, and the impedance of the receiver; or rather, just the sum r_2 of the latter two, which is supposed to be a simple resistance. We assume that the two instruments under consideration are absolutely identical, and connected by a long conductor. The resistance variations of the microphone are supposed to be small; the radian frequency is called ω ($= 2\pi \cdot$ frequency). It is difficult — even after the latest microphone investigations — to account for what actually happens when the resistance connected in series with a microphone is increased or decreased (unless the battery voltage is changed simultaneously so that the current remains constant); but we can evade this difficulty by supposing that the resistance of the primary winding is insignificant as compared to r_1; the resistance of the secondary is, similarly, considered small in proposition to r_2. It is also assumed that eddy-currents and hysteresis are negligible quantities. The last mentioned conditions are easily satisfied in practice if the available space is not too restricted, especially if it is possible to use thin core wires made of a good quality of iron. Now, we want to determine the two coefficients of self-induction, l_1 and l_2, and the coefficient of mutual induction, m; we will suppose that

$$\frac{l_2}{m} = \frac{m}{l_1}, \tag{1}$$

i. e. there is no leakage flux.

3. *Fundamental Equations.*

Under the above suppositions we can find the resultant impedance of the transformer by disconnecting the microphone and measuring the primary, or by disconnecting the receiver and conductor and measuring the secondary. We get, respectively,

$$R_1 = \frac{l_1 r_2}{\frac{r_2}{i\omega} + l_2} \tag{2}$$

$$R_2 = \frac{l_2 r_1}{\frac{r_1}{i\omega} + l_1}, \tag{3}$$

using the mathematical operator i in the now familiar manner. Furthermore we can find the alternating current values in the following circuits: the microphone circuit of the transmitting instrument; the telephone receiver of the latter, or the near end of the conductor; the far end of the conductor, or the telephone receiver of the receiving instrument; the microphone circuit of the receiving instrument. The last of these alternating current values is of no importance at all, however; the first three will be, respectively,

$$I_0 = \frac{Ed\left(\frac{r_2}{i\omega} + l_2\right)}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)} \tag{4}$$

$$I_1 = \frac{Edm}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)} \tag{5}$$

$$I_2 = \frac{2\,EdZe^{-\gamma l} m\left(\frac{r_1}{i\omega} + l_1\right)}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)^2} \tag{6}$$

It has been necessary to introduce some new denotations here, *viz.*

E = the battery e. m. f. or voltage
d = half of the variation of the microphone resistance during speech
Z = the characteristic impedance of the conductor
z = the numerical value of Z
l = the length of the conductor
γ = the propagation constant of the conductor $= \beta + i\alpha$
β = the real number component of γ.

In the case of the present investigation, however, all these quantities are constants, so that we may put

$$\frac{2\,Edze^{-\beta l}}{r_1} = k \tag{7}$$

Hence, considering especially I_2 (and only its numerical value, or amplitude, $|I_2|$, as the phase is of no importance in this respect), we obtain

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + l_1^2\right)}{\left((r_1 l_2 + r_2 l_1)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{8}$$

or, because $l_1 l_2 = m^2$,

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + l_1^2\right)}{\left(r_2^2 l_1^2 + \frac{r_1^2 m^4}{l_1^2} + 2 r_1 r_2 m^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{9}$$

or, by substituting L for l_1^2,

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(r_2^2 L + \frac{r_1^2 m^4}{L} + 2\, r_1 r_2 m^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{10}$$

4. First Condition To Be Satisfied.

If we, temporarily, let L be a constant, we must now — according to (10) — aim at securing a minimum value of the quantity

$$\left(r_2^2 L + \frac{r_1^2 r_2^2}{\omega^2}\right)\frac{1}{m} + 2\, r_1 r_2 m + \frac{r_1^2}{L}\, m^3;$$

we have, then,

$$\frac{3\, r_1^2}{L}\, m^2 + 2\, r_1 r_2 - \left(r_2^2\, L + \frac{r_1^2 r_2^2}{\omega^2}\right)\frac{1}{m^2} = 0, \tag{11}$$

or, by substituting M for m^2,

$$\frac{3\, r_1^2}{L} M^2 + 2\, r_1 r_2 M - \left(r_2^2\, L + \frac{r_1^2 r_2^2}{\omega^2}\right) = 0 \tag{12}$$

$$\frac{M}{L} = \frac{r_2}{3\, r_1}\left(-1 + \sqrt{4 + \frac{3\, r_1^2}{L\omega^2}}\right) \tag{13}$$

The value thus found will always be real and positive.

5. Second Condition To Be Satisfied.

Now let us regard $\frac{M}{L} = x$ as a constant. We have, from (8):

$$|I_2|^2 = k^2 \frac{M\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(L\left(r_1 \cdot \frac{M}{L} + r_2\right)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{14}$$

$$|I_2|^2 = k^2 \frac{xL\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(L\,(r_1 x + r_2)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{15}$$

The maximum value of this is obtained for

$$L = \frac{r_1^2}{\omega^2 \left(\frac{(r_1 x + r_2)^2}{r_2^2} - 2 \right)}, \tag{16a}$$

provided that $r_1 x > r_2 (\sqrt{2} - 1)$; if, on the other hand, $r_1 x < r_2 (\sqrt{2} - 1)$, it is necessary that

$$L = \infty, \tag{16b}$$

since L cannot be negative.

6. *Combined Result.*

It now remains to combine the results obtained in the two preceding sections. No serviceable result can be derived from (13) and (16a) taken together. (13) together with (16b) give the following result,

$$L = \infty \tag{17}$$

$$x = \frac{r_2}{3 r_1} \tag{18}$$

Here, the ratio n of the number of turns in the primary and secondary windings can be introduced; this ratio being the square root of x, we have

$$n = \sqrt{\frac{r_2}{3 r_1}}. \tag{19}$$

The corresponding maximum value of $|I_2|$ will be

$$|I_2|_{max} = \frac{3\sqrt{3}\, Edze^{-\beta l}}{8 r_1 r_2 \sqrt{r_1 r_2}}, \tag{20}$$

or

$$|I_2|_{max} = \frac{3\sqrt{3}}{16} k \cdot \frac{1}{r_2 \sqrt{r_1 r_2}} \tag{21}$$

The corresponding values of R_1 and R_2 will be

$$R_1 = 3 r_1 \tag{22}$$

$$R_2 = \tfrac{1}{3} r_2 \tag{23}$$

7. Comparison of Different Transformers.

It is often useful to have an algebraic expression for the serviceableness of different transformers. Then, a comparison of each individual transformer to the best possible one suggests itself naturally; the ratio of the two values of $|I_2|$ concerned being denoted by f, we arrive at the formula,

$$f = \frac{16}{3\sqrt{3}} \cdot \frac{\sqrt{x\left(\frac{r_2^2}{l_1^2\omega^2} + \frac{r_2^2}{r_1^2}\right)\cdot\frac{r_2}{r_1}}}{\left(x + \frac{r_2}{r_1}\right)^2 + \frac{r_2^2}{l_1^2\omega^2}}, \tag{24}$$

or

$$f = \frac{16}{3\sqrt{3}} \cdot \frac{n\sqrt{\left(\frac{r_2^2}{l_1^2\omega^2} + \frac{r_2^2}{r_1^2}\right)\frac{r_2}{r_1}}}{\left(n^2 + \frac{r_2}{r_1}\right)^2 + \frac{r_2^2}{l_1^2\omega^2}}. \tag{25}$$

8. Example.

Putting $r_1 = 20$; $r_2 = 2000$; $\omega = 5000$, we get a rather typical example, *viz.*

$$n = \sqrt{\frac{100}{3}} = 5 \cdot 8 \tag{26}$$

and furthermore, by means of (25), the following "double entry" table of f as a function of l_1 and n:

n \ l_1	·000	·001	·002	·003	·004	·005	·006	·007	·008	·009	·010	∞
5	·000	·362	·619	·768	·850	·896	·923	·939	·950	·958	·963	·989
6	·000	·427	·706	·849	·917	·950	·968	·978	·984	·988	·990	·999
7	·000	·488	·775	·898	·947	·965	·972	·975	·976	·976	·976	·971
8	·000	·544	·824	·919	·944	·948	·945	·941	·937	·934	·931	·916
9	·000	·593	·852	·914	·917	·907	·895	·886	·879	·873	·869	·846
10	·000	·635	·861	·888	·871	·850	·833	·820	·810	·803	·797	·770
11	·000	·669	·853	·848	·814	·786	·764	·749	·738	·730	·723	·694
12	·000	·694	·830	·797	·752	·718	·694	·678	·666	·657	·651	·621
13	·000	·710	·797	·740	·688	·651	·627	·610	·598	·589	·583	·553
14	·000	·718	·756	·682	·625	·587	·563	·547	·535	·527	·520	·492
15	·000	·717	·709	·624	·565	·528	·504	·489	·478	·470	·464	·437

Such a table can be utilized in various ways. Thus, values taken along vertical or horizontal lines may be used for plotting curves; or l_1 and n may be regarded as right-angled coordinates for points in a plane, and so curves can be plotted in this plane corresponding to $f = 0{\cdot}90$, $0{\cdot}80$, &c. In practice, all transformers belonging on the proper side of the curve for $f = 0{\cdot}90$ will probably not be much inferior to the theoretically correct one. — The reasons why the curves mentioned are not shown here, are that they are so easy to plot, and that the table chiefly gives the same information.

9. *Articulation.*

Apart from $\omega = 5000$, other somewhat smaller and larger values must be considered, if not only a high power level, but also high intelligibility in the transmission of speech is desired. For this purpose, however, no special calculation is necessary; as formula (25) shows, a change of ω can always be regarded as equivalent to a change of l_1, and the desired values of f as corresponding to different values of ω are consequently to be found in a horizontal line of the above mentioned table (or from the correspondingly derived curve). Here, only the articulation reduction due to the transformer has been taken into consideration (and r_2 is accordingly regarded as being constant for the different frequencies); as it happens, this reduction is generally of no importance as compared to that due to the variation of the β of the cables in particular, and perhaps — to some extent — to the variation of the d of the microphone.

10. *Resistance of Transmitter and Receiver.*

The formula (20) gives, at least, some information as to the question of what resistance values should be chosen for the microphone and the telephone receiver, although it is an indication only — not a directly applicable rule — since neither of these is so simple a device as the transformer. Let the best telephone be the one that receives the most energy; we shall then find that the resistance of the telephone should be about $\frac{1}{3} r_2$, or about $\frac{1}{2} z$. Regarding the microphone we may work on the hypothesis that d and r_1 are proportionals; there seems to be reason, also, for assuming that the battery voltage and the square root of r_1 should be proportionals, *e. g.* with a view to the necessary cooling of the microphone. Under these suppositions, all values of r_1 may prove to be equally good.

11. Inductive Shunt.

It seems natural to ask whether some other device might not be just as useful, or even more so, than a transformer; it would rather have to be a low-resistance shunt with a high coefficient of self-induction. The transformer which, according to the above, was selected as the best would then, for $r_2 = 3\ r_1$, be equivalent to the shunt. In practice, however, we always have $r_2 > 3\ r_1$, and the transformer is therefore to be preferred.

12. Conclusion.

We have in the foregoing been dealing with the Local Battery instrument, under presuppositions chosen so as to make the calculations fairly simple, but without departing essentially from what is usual or possible in practice. The supposition that the impedance r_2 is a simple resistance may, of course, be abandoned, and an arbitrary impedance taken instead; but there is scarcely anything to be gained by that. In the essentials, the problem in connexion with Central Battery instruments is the same as above, although the presuppositions should, perhaps, be chosen somewhat differently; besides, the transformer is not quite so important here as in the case of the L. B. instrument.

11. NEW ALTERNATING-CURRENT COMPENSATION APPARATUS FOR TELEPHONIC MEASUREMENTS

First published in "Elektroteknikeren", vol. 9, 1913, p. 157.

Description of the Apparatus.

The essential parts of the apparatus are (see Fig. 1): Two calibrated slide-wires M_1 and M_2 (each having a resistance of about 50 ohms) with a millimetre graduation in both directions from a central zero; two standards of self-induction P, each of 0.01 henry and 3 ohms; two non-

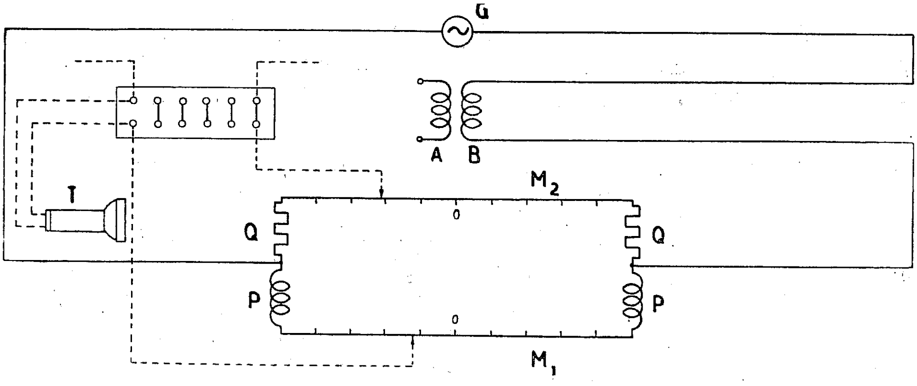

Fig. 1.

inductive coils, Q, of exactly the same resistance as the standards; a transformer, containing no iron, and consisting of a primary coil B and a secondary coil A, the mutual inductance being variable within certain limits by moving one of the coils. Two sliding contacts, each movable along one of the wires M_1 and M_2, are during the measurements generally connected with different points on the object to be measured (which is not shown in the illustration); these connexions (and others which are found desirable later) are facilitated by a junction-board having a number of terminals. A telephone receiver T, the cords of which are led to this junction-board, forms part of one of the above connexions.

The Pressure Diagram.

When a simple periodic alternating current flows through the two parallel circuits consisting of the coils P with the wire M_1 and the coils Q with the wire M_2, the pressure along the wires can be represented by means of a vector diagram, as in Fig. 2. The numbers shown on this diagram correspond directly with the marking of the two scales of the apparatus.

Knowing the frequency ω of the current, and the constants of the wires and coils, the angle α between the two lines can easily be determined. Taking, for instance, the values given previously, we have —

$$\omega = 2{,}800 \tan \alpha.$$

To check the accuracy of the wires, &c., the two junction-terminals used for connecting up to the object to be measured can be short-circuited; in that case, on placing the sliding contacts at zero, the telephone should be silent.

The Generator and the Frequency.

Any alternating-current generator capable of producing currents of the desired frequency or frequencies (for instance $\omega = 5{,}000$) may be used, provided it gives a wave sufficiently free from harmonics. Its output need not be large or very constant, but it is important that the frequency should remain practically constant during each measurement.

For determining the frequency, two methods have been found very convenient. First, use can be made of a tuning-fork of the desired frequency of vibration by comparing the pitches and observing the beats. The second method is purely electrical. By connecting the terminals of the secondary coil A to the junction-board, it is possible by moving the sliding contacts to balance the secondary pressure against the potential difference between two points of the wires. Having obtained silence in the telephone, the two readings a and b are noted. It is, by the way, advisable here to reverse the connexions and to take a second set of readings, which ought to be equal to the first with the signs reversed. A similar check may be applied to all the following measurements. Now it can be shown[1]) that

$$\tan \alpha = \sqrt{2\,(b/a - 1)},$$

and by using the formula for ω given above, ω can easily be found. It must be realized that the ratio b/a will be independent of the distance between the coils A and B; on the other hand, if the same distance is

[1]) The proof of this must here be omitted.

always chosen the reading "a" will be the same for every value of ω, and the frequency can accordingly be measured rather more easily.

Impedance Measurements.

The unknown impedance is connected, together with a known resistance, say, 1,000 ohms, in series with the secondary coil. Now, by compensation, the pressures across the known resistance and across the unknown imped-

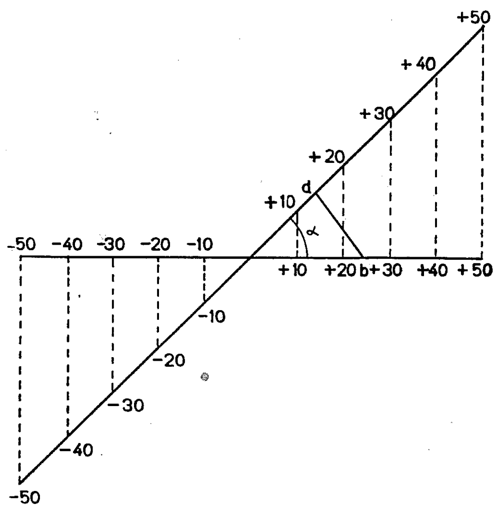

Fig. 2.

ance are ascertained, the corresponding vectors being at once determined from the pressure diagram (Fig. 2). From the lengths of the vectors and the angle between them, the vector representing the unknown impedance (the ohm being taken as the unit) can easily be found.

Transmission Measurements on Telephone Conductors.

For the sake of simplicity only homogeneous conductors will be considered. One method is to measure the impedance (A and K) of the conductor with the far end first open and then closed. The constant, z, known as the "characteristic impedance" or "initial sending-end impedance", can then be found as the geometrical mean of A and K.

The propagation constants, generally denoted by β and α, are most easily found by constructing a triangle, the two sides of which, A and K, enclose the correct angle between them. The third side will be C (Fig. 3). Now, denoting by l the length of the conductor, we have —

$$\cosh\ 2\,\beta\,l = \frac{A + K}{C},$$

$$\cos\ 2\,\alpha\,l = \frac{A - K}{C}.$$

In these equations, A, K, and C denote the absolute values of the vector quantities. Of course, in order to determine $\alpha\,l$ it is necessary to have a preliminary approximate knowledge of its value.

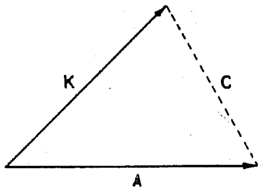

Fig. 3.

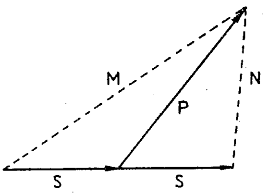

Fig. 4.

Another method of finding β and α consists in measuring the potential differences, P and S, at the near end and at the far end — it must here be assumed that the far end of the conductor can be brought within reach. For obtaining the values of β and α the construction shown in Fig. 4 may be used; we then have —

$$\cosh\,\beta\,l = \frac{M + N}{2\,S}$$

$$\cos\,\alpha\,l = \frac{M - N}{2\,S},$$

M, N, and S being the absolute values of the vector quantities.

Generally speaking, from the point of view of accuracy the latter method is preferable if the conductor to be measured is long, the former if it is short.

Instead of the potential difference, S, it is sometimes better to measure the fall of pressure from the near end to the far end.

In the case of a non-homogeneous conductor, the methods are not quite so simple as in the case just considered. A complete measurement consists of 3 impedance measurements, one of which can be omitted by measuring the pressure at the far end.

The case of a transformer is analogous.

Special Methods.

While the above-mentioned methods are sufficient for the telephone measurements most commonly required in the laboratory or in practice, a somewhat different arrangement of the main parts of the apparatus

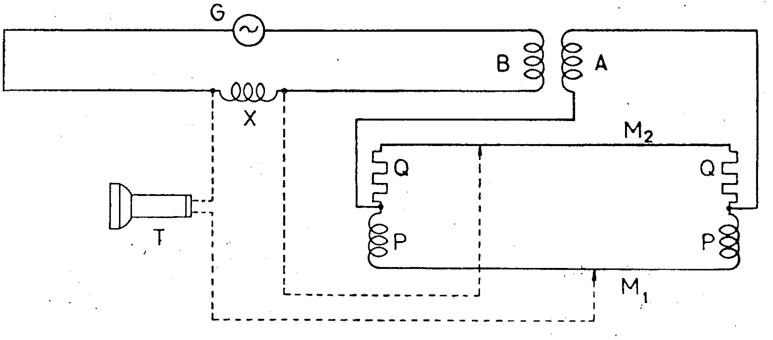

Fig. 5.

will occasionally be more convenient. Thus for impedance measurements the arrangements indicated in Figs. 5 and 6 can be used under certain circumstances, the former for not too great, the latter for not too small impedances. It is here unnecessary to have a known comparison resistance in series with the impedance X to be determined, provided that once for

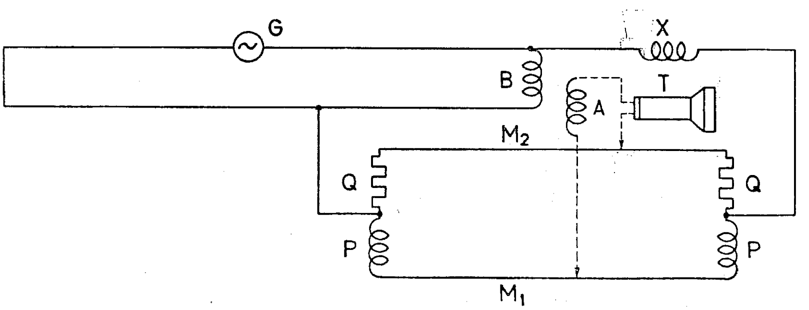

Fig. 6.

all such a resistance is inserted in place of X and the corresponding readings are taken.

The arrangement shown in Fig. 7 is very convenient for determinations of the propagation constants β and α[1]) (the near-end pressure may be measured once for all, it being the same in all cases).

[1]) β denotes the attenuation constant and α the phase constant.

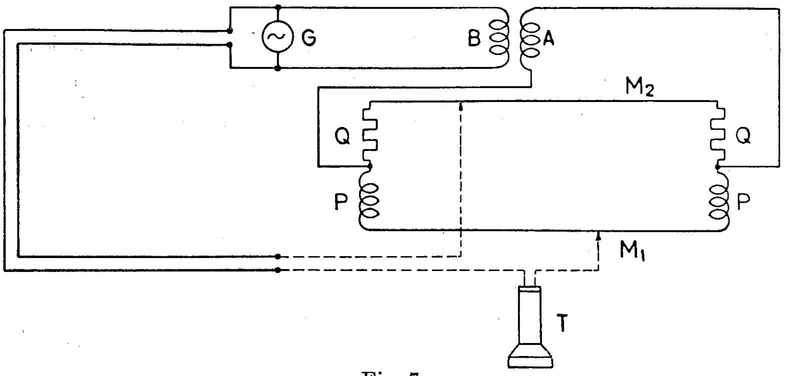

Fig. 7.

The arrangement indicated in Fig. 8 is suitable for microphone tests, and some difficulty will naturally be experienced owing to the somewhat irregular behaviour of the microphone.

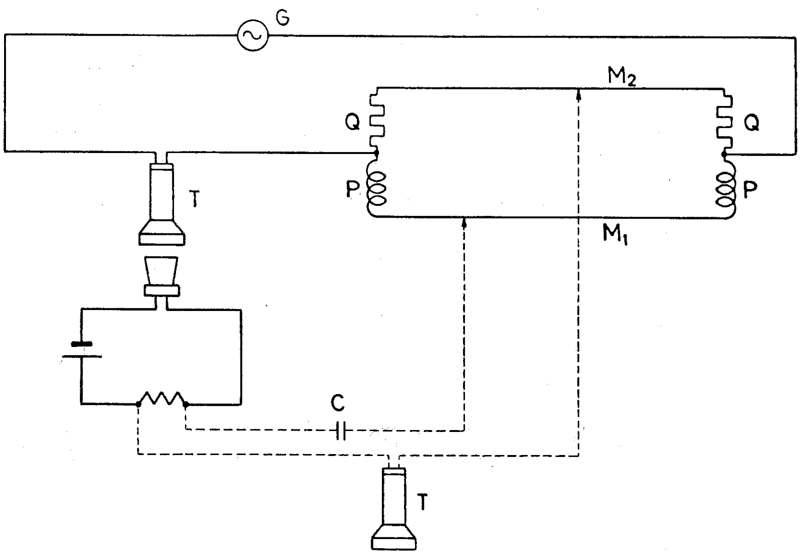

Fig. 8.

Conclusion.

A description has here been given of a simple, cheap, and transportable apparatus for telephonic and other alternating-current measurements, which has already done good service for some years in the research laboratory of the Copenhagen Telephone Company, and which may prove of interest to others handling similar problems.

In conclusion, it is a pleasant duty to express my grateful acknowledgments of several very valuable researches already made on similar lines, especially the work of Professor *Absalon Larsen* of Copenhagen[1]). My special thanks are also due to Mr. *J. L. W. V. Jensen*, Chief Engineer of the Copenhagen Telephone Company, under whose supervision the present work has been carried out.

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[1]) I refer to the following two papers by Professor Larsen in the *Elektrotechnische Zeitschrift*: "Ein akustischer Wechselstromerzeuger mit regulierbarer Periodenzahl für schwache Ströme" (vol. 32, p. 284, 1911), and "Der komplexe Kompensator, ein Apparat zur Messung von Wechselströmen durch Kompensation" (vol. 31, p. 1039, 1910).

TABLE OF ERLANG'S LOSS FORMULA

Computed by E. BROCKMEYER

$$\text{Formula: } B = \frac{\frac{y^n}{n!}}{1 + y + \frac{y^2}{2!} + \cdots + \frac{y^n}{n!}}$$

where B is the loss, n the number of circuits and y the traffic-intensity in erlang.

The table gives for $n = 1 — 260$ the values of y corresponding to various fixed values of B.

n	B=0.001	B=0.002	B=0.003	B=0.004	B=0.005	B=0.010	B=0.020	B=0.050
1	0.001	0.002	0.003	0.004	0.005	0.01	0.02	0.05
2	0.05	0.07	0.08	0.09	0.11	0.15	0.22	0.38
3	0.19	0.25	0.29	0.32	0.35	0.46	0.60	0.90
4	0.44	0.53	0.60	0.66	0.70	0.87	1.09	1.52
5	0.76	0.90	0.99	1.07	1.13	1.36	1.66	2.22
6	1.15	1.33	1.45	1.54	1.62	1.91	2.28	2.96
7	1.58	1.80	1.95	2.06	2.16	2.50	2.94	3.74
8	2.05	2.31	2.48	2.62	2.73	3.13	3.63	4.54
9	2.56	2.85	3.05	3.21	3.33	3.78	4.34	5.37
10	3.09	3.43	3.65	3.82	3.96	4.46	5.08	6.22
11	3.65	4.02	4.26	4.45	4.61	5.16	5.84	7.08
12	4.23	4.64	4.90	5.11	5.28	5.88	6.62	7.95
13	4.83	5.27	5.56	5.78	5.96	6.61	7.41	8.83
14	5.45	5.92	6.23	6.47	6.66	7.35	8.20	9.73
15	6.08	6.58	6.91	7.17	7.38	8.11	9.01	10.63

n	B=0.001	B=0.002	B=0.003	B=0.004	B=0.005	B=0.010	B=0.020	B=0.050
16	6.72	7.26	7.61	7.88	8.10	8.87	9.83	11.54
17	7.38	7.95	8.32	8.60	8.83	9.65	10.66	12.46
18	8.05	8.64	9.03	9.33	9.58	10.44	11.49	13.38
19	8.72	9.35	9.76	10.07	10.33	11.23	12.33	14.31
20	9.41	10.07	10.50	10.82	11.09	12.03	13.18	15.25
21	10.11	10.79	11.24	11.58	11.86	12.84	14.04	16.19
22	10.81	11.53	11.99	12.34	12.63	13.65	14.90	17.13
23	11.52	12.27	12.75	13.11	13.42	14.47	15.76	18.08
24	12.24	13.01	13.51	13.89	14.20	15.29	16.63	19.03
25	12.97	13.76	14.28	14.67	15.00	16.12	17.50	19.99
26	13.70	14.52	15.05	15.46	15.80	16.96	18.38	20.94
27	14.44	15.28	15.83	16.25	16.60	17.80	19.26	21.90
28	15.18	16.05	16.62	17.05	17.41	18.64	20.15	22.87
29	15.93	16.83	17.41	17.85	18.22	19.49	21.04	23.83
30	16.68	17.61	18.20	18.66	19.03	20.34	21.93	24.80
31	17.44	18.39	19.00	19.47	19.85	21.19	22.83	25.77
32	18.20	19.18	19.80	20.28	20.68	22.05	23.73	26.75
33	18.97	19.97	20.61	21.10	21.51	22.91	24.63	27.72
34	19.74	20.76	21.42	21.92	22.34	23.77	25.53	28.70
35	20.52	21.56	22.23	22.75	23.17	24.64	26.43	29.68
36	21.30	22.36	23.05	23.58	24.01	25.51	27.34	30.66
37	22.08	23.17	23.87	24.41	24.85	26.38	28.25	31.64
38	22.86	23.97	24.69	25.24	25.69	27.25	29.17	32.63
39	23.65	24.78	25.52	26.08	26.53	28.13	30.08	33.61
40	24.44	25.60	26.35	26.92	27.38	29.01	31.00	34.60
41	25.24	26.42	27.18	27.76	28.23	29.89	31.92	35.59
42	26.04	27.24	28.01	28.60	29.08	30.77	32.84	36.58
43	26.84	28.06	28.85	29.45	29.94	31.66	33.76	37.57
44	27.64	28.88	29.68	30.30	30.80	32.54	34.68	38.56
45	28.45	29.71	30.52	31.15	31.66	33.43	35.61	39.55
46	29.26	30.54	31.37	32.00	32.52	34.32	36.53	40.54
47	30.07	31.37	32.21	32.85	33.38	35.21	37.46	41.54
48	30.88	32.20	33.06	33.71	34.25	36.11	38.39	42.54
49	31.69	33.04	33.91	34.57	35.11	37.00	39.32	43.54
50	32.51	33.88	34.76	35.43	35.98	37.90	40.25	44.53

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
51	33.33	34.72	35.61	36.29	36.85	38.80
52	34.15	35.56	36.47	37.16	37.72	39.70
53	34.98	36.40	37.32	38.02	38.60	40.60
54	35.80	37.25	38.18	38.89	39.47	41.50
55	36.63	38.09	39.04	39.76	40.35	42.41
56	37.46	38.94	39.90	40.63	41.23	43.31
57	38.29	39.79	40.76	41.50	42.11	44.22
58	39.12	40.64	41.63	42.38	42.99	45.13
59	39.96	41.50	42.49	43.25	43.87	46.04
60	40.79	42.35	43.36	44.13	44.76	46.95
61	41.63	43.21	44.23	45.00	45.64	47.86
62	42.47	44.07	45.10	45.88	46.53	48.77
63	43.31	44.93	45.97	46.76	47.42	49.69
64	44.16	45.79	46.84	47.64	48.31	50.60
65	45.00	46.65	47.72	48.53	49.20	51.52
66	45.84	47.51	48.59	49.41	50.09	52.44
67	46.69	48.38	49.47	50.30	50.98	53.35
68	47.54	49.24	50.34	51.18	51.87	54.27
69	48.39	50.11	51.22	52.07	52.77	55.19
70	49.24	50.98	52.10	52.96	53.66	56.11
71	50.09	51.85	52.98	53.85	54.56	57.03
72	50.94	52.72	53.87	54.74	55.45	57.96
73	51.80	53.59	54.75	55.63	56.35	58.88
74	52.65	54.46	55.63	56.52	57.25	59.80
75	53.51	55.34	56.52	57.42	58.15	60.73
76	54.37	56.21	57.40	58.31	59.05	61.65
77	55.23	57.09	58.29	59.21	59.96	62.58
78	56.09	57.96	59.18	60.10	60.86	63.51
79	56.95	58.84	60.07	61.00	61.76	64.43
80	57.81	59.72	60.96	61.90	62.67	65.36
81	58.67	60.60	61.85	62.80	63.57	66.29
82	59.54	61.48	62.74	63.69	64.48	67.22
83	60.40	62.36	63.63	64.59	65.38	68.15
84	61.27	63.24	64.52	65.50	66.29	69.08
85	62.14	64.13	65.41	66.40	67.20	70.02

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
86	63.00	65.01	66.31	67.30	68.11	70.95
87	63.87	65.90	67.20	68.20	69.02	71.88
88	64.74	66.78	68.10	69.11	69.93	72.81
89	65.61	67.67	69.00	70.01	70.84	73.75
90	66.48	68.56	69.90	70.92	71.76	74.68
91	67.36	69.44	70.79	71.82	72.67	75.62
92	68.23	70.33	71.69	72.73	73.58	76.56
93	69.10	71.22	72.59	73.64	74.49	77.49
94	69.98	72.11	73.49	74.55	75.41	78.43
95	70.85	73.00	74.40	75.45	76.32	79.37
96	71.73	73.90	75.30	76.36	77.24	80.31
97	72.61	74.79	76.20	77.27	78.16	81.24
98	73.48	75.68	77.10	78.18	79.07	82.18
99	74.36	76.57	78.01	79.10	79.99	83.12
100	75.24	77.47	78.91	80.01	80.91	84.06
101	76.12	78.36	79.82	80.92	81.83	85.00
102	77.00	79.26	80.72	81.83	82.75	85.95
103	77.88	80.16	81.63	82.75	83.67	86.89
104	78.77	81.05	82.53	83.66	84.59	87.83
105	79.65	81.95	83.44	84.58	85.51	88.77
106	80.53	82.85	84.35	85.49	86.43	89.72
107	81.42	83.75	85.26	86.41	87.35	90.66
108	82.30	84.65	86.17	87.32	88.27	91.60
109	83.19	85.55	87.08	88.24	89.20	92.55
110	84.07	86.45	87.99	89.16	90.12	93.49
111	84.96	87.35	88.90	90.08	91.04	94.44
112	85.85	88.25	89.81	90.99	91.97	95.38
113	86.73	89.15	90.72	91.91	92.89	96.33
114	87.62	90.06	91.63	92.83	93.82	97.28
115	88.51	90.96	92.54	93.75	94.74	98.22
116	89.40	91.86	93.46	94.67	95.67	99.17
117	90.29	92.77	94.37	95.59	96.60	100.12
118	91.18	93.67	95.28	96.51	97.53	101.07
119	92.07	94.58	96.20	97.44	98.45	102.02
120	92.96	95.48	97.11	98.36	99.38	102.96

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
121	93.86	96.39	98.03	99.28	100.31	103.91
122	94.75	97.30	98.95	100.20	101.24	104.86
123	95.64	98.20	99.86	101.13	102.17	105.81
124	96.54	99.11	100.78	102.05	103.10	106.76
125	97.43	100.02	101.70	102.98	104.03	107.71
126	98.33	100.93	102.61	103.90	104.96	108.66
127	99.22	101.84	103.53	104.83	105.89	109.62
128	100.12	102.75	104.45	105.75	106.82	110.57
129	101.01	103.66	105.37	106.68	107.75	111.52
130	101.91	104.57	106.29	107.60	108.68	112.47
131	102.81	105.48	107.21	108.53	109.62	113.42
132	103.70	106.39	108.13	109.46	110.55	114.38
133	104.60	107.30	109.05	110.38	111.48	115.33
134	105.50	108.22	109.97	111.31	112.42	116.28
135	106.40	109.13	110.89	112.24	113.35	117.24
136	107.30	110.04	111.82	113.17	114.28	118.19
137	108.20	110.95	112.74	114.10	115.22	119.14
138	109.10	111.87	113.66	115.03	116.15	120.10
139	110.00	112.78	114.58	115.96	117.09	121.05
140	110.90	113.70	115.51	116.89	118.02	122.01
141	111.81	114.61	116.43	117.82	118.96	122.96
142	112.71	115.53	117.36	118.75	119.90	123.92
143	113.61	116.44	118.28	119.68	120.83	124.88
144	114.51	117.36	119.20	120.61	121.77	125.83
145	115.42	118.28	120.13	121.54	122.71	126.79
146	116.32	119.19	121.05	122.47	123.64	127.74
147	117.22	120.11	121.98	123.41	124.58	128.70
148	118.13	121.03	122.91	124.34	125.52	129.66
149	119.03	121.95	123.83	125.27	126.46	130.62
150	119.94	122.86	124.76	126.21	127.40	131.58
151	120.85	123.78	125.69	127.14	128.33	132.53
152	121.75	124.70	126.61	128.07	129.27	133.49
153	122.66	125.62	127.54	129.01	130.21	134.45
154	123.56	126.54	128.47	129.94	131.15	135.41
155	124.47	127.46	129.40	130.88	132.09	136.37

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
156	125.38	128.38	130.33	131.81	133.03	137.33
157	126.29	129.30	131.25	132.75	133.97	138.29
158	127.20	130.22	132.18	133.68	134.91	139.25
159	128.10	131.14	133.11	134.62	135.86	140.21
160	129.01	132.07	134.04	135.55	136.80	141.17
161	129.92	132.99	134.97	136.49	137.74	142.13
162	130.83	133.91	135.90	137.43	138.68	143.09
163	131.74	134.83	136.83	138.36	139.62	144.05
164	132.65	135.75	137.76	139.30	140.56	145.01
165	133.56	136.68	138.70	140.24	141.51	145.97
166	134.47	137.60	139.63	141.18	142.45	146.93
167	135.39	138.52	140.56	142.11	143.39	147.89
168	136.30	139.45	141.49	143.05	144.34	148.86
169	137.21	140.37	142.42	143.99	145.28	149.82
170	138.12	141.30	143.36	144.93	146.22	150.78
171	139.03	142.22	144.29	145.87	147.17	151.74
172	139.95	143.15	145.22	146.81	148.11	152.71
173	140.86	144.07	146.16	147.75	149.06	153.67
174	141.77	145.00	147.09	148.69	150.00	154.63
175	142.69	145.92	148.02	149.63	150.95	155.60
176	143.60	146.85	148.96	150.57	151.89	156.56
177	144.52	147.78	149.89	151.51	152.84	157.52
178	145.43	148.70	150.83	152.45	153.78	158.49
179	146.35	149.63	151.76	153.39	154.73	159.45
180	147.26	150.56	152.70	154.33	155.68	160.42
181	148.18	151.49	153.63	155.27	156.62	161.38
182	149.09	152.41	154.57	156.21	157.57	162.35
183	150.01	153.34	155.50	157.16	158.52	163.31
184	150.93	154.27	156.44	158.10	159.46	164.28
185	151.84	155.20	157.37	159.04	160.41	165.24
186	152.76	156.13	158.31	159.98	161.36	166.21
187	153.68	157.06	159.25	160.93	162.31	167.17
188	154.59	157.99	160.19	161.87	163.25	168.14
189	155.51	158.91	161.12	162.81	164.20	169.10
190	156.43	159.84	162.06	163.76	165.15	170.07

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
191	157.35	160.77	163.00	164.70	166.10	171.03
192	158.27	161.70	163.94	165.64	167.05	172.00
193	159.19	162.64	164.87	166.59	168.00	172.97
194	160.10	163.57	165.81	167.53	168.95	173.93
195	161.02	164.50	166.75	168.47	169.90	174.90
196	161.94	165.43	167.69	169.42	170.84	175.87
197	162.86	166.36	168.63	170.36	171.79	176.84
198	163.78	167.29	169.57	171.31	172.74	177.80
199	164.70	168.22	170.51	172.25	173.69	178.77
200	165.62	169.16	171.45	173.20	174.64	179.74
201	166.54	170.09	172.39	174.15	175.60	180.71
202	167.47	171.02	173.33	175.09	176.55	181.67
203	168.39	171.95	174.27	176.04	177.50	182.64
204	169.31	172.88	175.21	176.98	178.45	183.61
205	170.23	173.82	176.15	177.93	179.40	184.58
206	171.15	174.75	177.09	178.88	180.35	185.55
207	172.07	175.68	178.03	179.82	181.30	186.52
208	173.00	176.62	178.97	180.77	182.25	187.48
209	173.92	177.55	179.91	181.72	183.21	188.45
210	174.84	178.49	180.85	182.66	184.16	189.42
211	175.77	179.42	181.79	183.61	185.11	190.39
212	176.69	180.36	182.74	184.56	186.06	191.36
213	177.61	181.29	183.68	185.51	187.01	192.33
214	178.54	182.23	184.62	186.45	187.97	193.30
215	179.46	183.16	185.56	187.40	188.92	194.27
216	180.38	184.10	186.51	188.35	189.87	195.24
217	181.31	185.03	187.45	189.30	190.83	196.21
218	182.23	185.97	188.39	190.25	191.78	197.18
219	183.16	186.90	189.33	191.20	192.73	198.15
220	184.08	187.84	190.28	192.15	193.69	199.12
221	185.01	188.78	191.22	193.09	194.64	200.09
222	185.93	189.71	192.16	194.04	195.59	201.06
223	186.86	190.65	193.11	194.99	196.55	202.04
224	187.78	191.58	194.05	195.94	197.50	203.01
225	188.71	192.52	195.00	196.89	198.46	203.98

n	$B = 0.001$	$B = 0.002$	$B = 0.003$	$B = 0.004$	$B = 0.005$	$B = 0.010$
226	189.64	193.46	195.94	197.84	199.41	204.95
227	190.56	194.40	196.89	198.79	200.37	205.92
228	191.49	195.33	197.83	199.74	201.32	206.89
229	192.42	196.27	198.78	200.69	202.28	207.86
230	193.34	197.21	199.72	201.64	203.23	208.84
231	194.27	198.15	200.67	202.60	204.19	209.81
232	195.20	199.09	201.61	203.55	205.14	210.78
233	196.13	200.02	202.56	204.50	206.10	211.75
234	197.05	200.96	203.50	205.45	207.05	212.72
235	197.98	201.90	204.45	206.40	208.01	213.70
236	198.91	202.84	205.39	207.35	208.97	214.67
237	199.84	203.78	206.34	208.30	209.92	215.64
238	200.77	204.72	207.29	209.26	210.88	216.62
239	201.70	205.66	208.23	210.21	211.83	217.59
240	202.62	206.60	209.18	211.16	212.79	218.56
241	203.55	207.54	210.13	212.11	213.75	219.53
242	204.48	208.48	211.07	213.06	214.70	220.51
243	205.41	209.42	212.02	214.02	215.66	221.48
244	206.34	210.36	212.97	214.97	216.62	222.45
245	207.27	211.30	213.92	215.92	217.58	223.43
246	208.20	212.24	214.86	216.87	218.53	224.40
247	209.13	213.18	215.81	217.83	219.49	225.37
248	210.06	214.12	216.76	218.78	220.45	226.35
249	210.99	215.06	217.71	219.73	221.41	227.32
250	211.92	216.00	218.65	220.69	222.36	228.30
251	212.85	216.94	219.60	221.64	223.32	229.27
252	213.78	217.88	220.55	222.59	224.28	230.25
253	214.72	218.83	221.50	223.55	225.24	231.22
254	215.65	219.77	222.45	224.50	226.20	232.19
255	216.58	220.71	223.40	225.46	227.16	233.17
256	217.51	221.65	224.34	226.41	228.11	234.14
257	218.44	222.59	225.29	227.37	229.07	235.12
258	219.37	223.54	226.24	228.32	230.03	236.09
259	220.31	224.48	227.19	229.27	230.99	237.07
260	221.24	225.42	228.14	230.23	231.95	238.04

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